$\sqrt{bc}$ Inversion

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Thank you, Evan Chen, for this beautiful style package!

§1 Revision of inversion

Before we start exploring $\sqrt{bc}$ inversion, let's recall some important facts about ordinary inversion starting with the definition.

> Let Ω be a circle with center O and radius r. We say that inversion about Ω is a map, which maps any point P in the plane (except for O) to a point P' (we say that P' is the **inverse** of P) on ray OP such that $OP \cdot OP' = r^2$. It is easy to see that the inverse of P' is P.

To deal with the center of the circle O, we will add a new point to the plane called P_∞ – a **point at infinity**, which every line passes through (and no circle passes through). Then we say that inversion about Ω swaps points P_∞ and O.

The power of inversion lies in the fact that it allows us to transform one problem to an equivalent one, which sometimes may be a lot easier to solve. The most important properties of inversion are the following:

- A line passing through O maps to itself
- A line not passing through O maps to a circle passing through O (and the converse is also true!)
- A circle not passing through O maps to a circle not passing through O
- If A and B are points other than O, then $A'B' = \dfrac{r^2}{OA \cdot OB} \cdot AB$
- $\angle OAB = \angle OB'A'$

Notice that the number of intersections of two objects remains the same after applying inversion, which leads us to an important claim that inversion **preserves tangencies**.

We will also make use of the following important lemma, which is a consequence of the proof of the last bullet point:

> **Lemma 1.1** (Collinearity of circumcenters under inversion)
> If a circle ω not passing through O gets mapped to a circle γ, then O and the circumcenters of ω and γ are collinear.

Now let's brush up on our inversion problem solving skills by solving some warm-up problems!

§2 Inversion warm-up

Problem 2.1 (Archimedes' Lemma)**.** Let BC be a chord of a circle Ω. Let ω be a circle tangent to chord BC at K and internally tangent to ω at T. Then ray TK passes through the midpoint of the arc $\widehat{BC}$ not containing T.

Problem 2.2 (USAMO 1993/2)**.** Let $ABCD$ be a quadrilateral whose diagonals AC and BD are perpendicular and intersect at E. Prove that the reflections E across AB, BC, CD, DA are concyclic.

Problem 2.3. Let A, B and C be three collinear points and suppose P is any point in the plane. Prove that the circumcenters of triangles PAB, PAC, PBC and P are concyclic.

§3 $\sqrt{bc}$ Inversion

We are now ready to introduce the concept of $\sqrt{bc}$ inversion: a special transformation, which combines inversion with reflection across the angle bisector.

Lemma ($\sqrt{bc}$ inversion)

Let triangle ABC be given. $\sqrt{bc}$ inversion is a composition of two familiar maps:

- Firstly, we invert at A with radius $\sqrt{AB \cdot AC}$
- Secondly, we perform a reflection across the angle bisector of $\angle BAC$

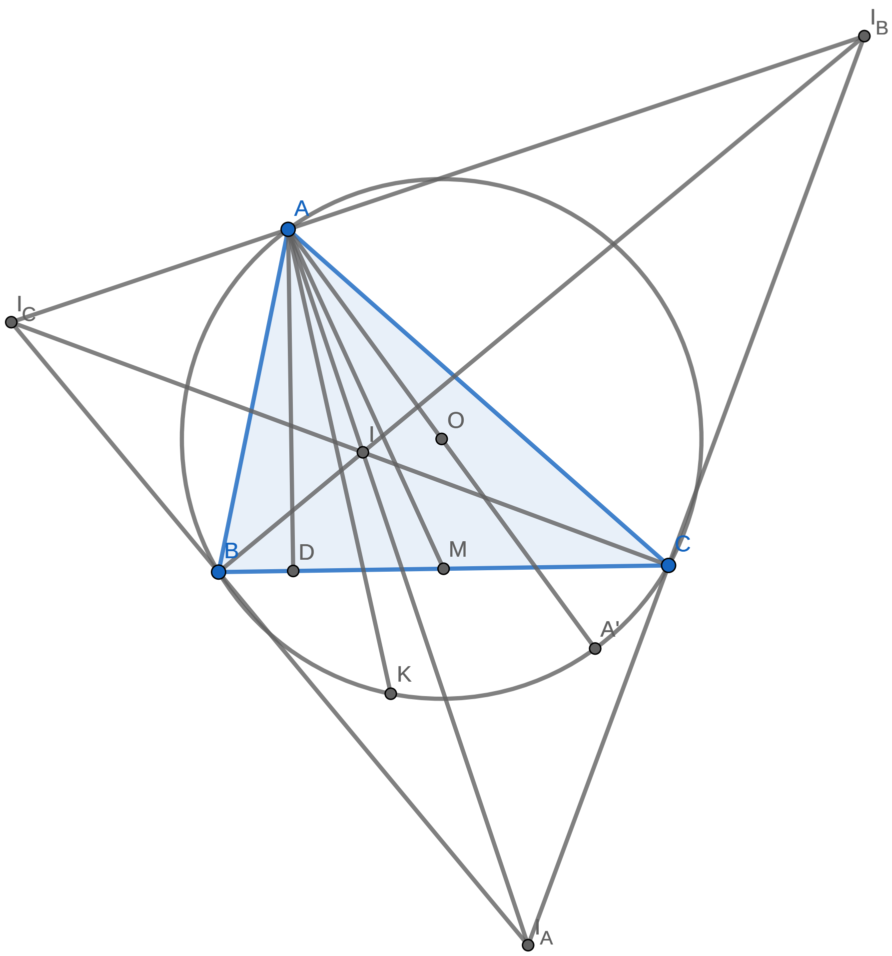
The concept of $\sqrt{bc}$ inversion is particularly useful when there are isogonal lines involved in the problem, because isogonal lines will swap with each other.

§3.1 Where does $\sqrt{bc}$ inversion map familiar points?

Let's now investigate where do all the important points and objects go under this transformation.

Claim — Let $\triangle ABC$ be given. Denote by I, O, H its incenter, circumcenter and orthocenter, respectively. Let I_A, I_B and I_C be its excenters. Also, let A' be the A antipode with respect to (ABC) and D be the foot of a perpendicular point A to the line BC. We have that:

- B swaps with C. Thus, BC swaps with (ABC)
- I swaps with I_A
- I_B swaps with I_C
- D swaps with A'
- O swaps with the reflection of A across BC
- If K is the intersection of A-symmedian with (ABC) and M is the midpoint of the segment BC, then M swaps with K



§3.2 Mixtillinear incircle and excircle

Let triangle ABC be given, then A-**mixtillinear incircle** is a circle, which is tangent to segments AB and AC and also the circumcircle of $\triangle ABC$.

We can similarly define the A-mixtillinear excircle: it is a circle, which is tangent to **lines** AB and AC and also externally to the circumcircle of $\triangle ABC$. $\sqrt{bc}$ inversion acts very nicely on mixtillinear circles, because it sends them to some very familiar objects. Here are some facts about mixtillinear circles that can be proved using $\sqrt{bc}$ inversion:

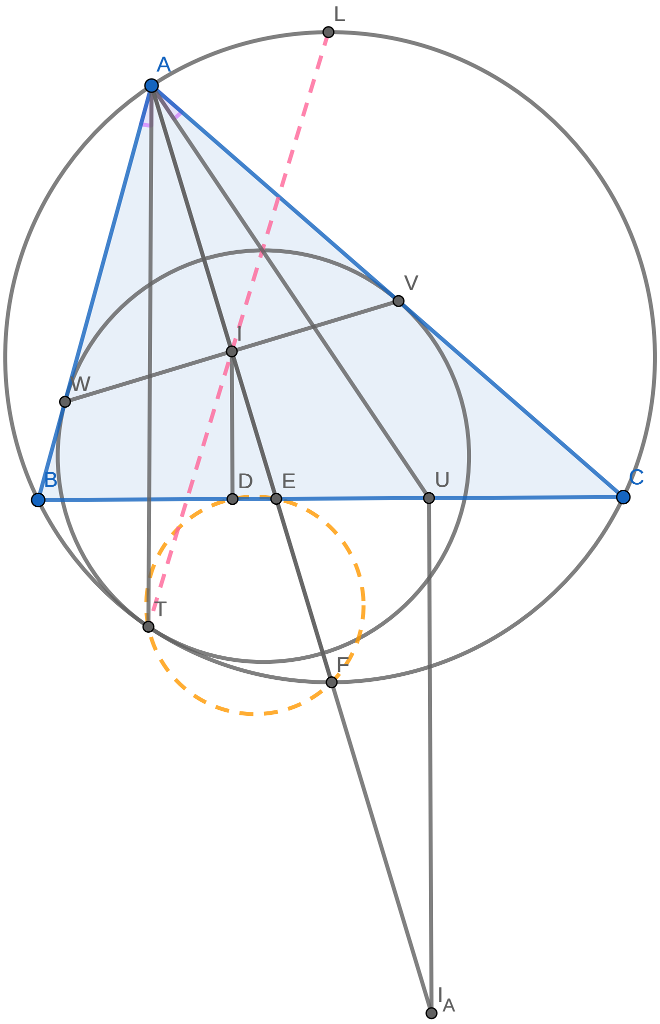
Claim — Let T be the touchpoint of (ABC) and A-mixtillinear incircle, L be the midpoint of arc BAC of (ABC) and U be the touchpoint of A-excircle with BC. Then we have the following:

- The A-mixtillinear incircle swaps with A-excircle
- The A-mixtillinear excircle swaps with the incircle of $\triangle ABC$
- $\angle BAT = \angle CAU$
- Points L, I and T are collinear

There are also some very important properties of the mixtillinear incircle that cannot be proved conveniently with $\sqrt{bc}$ inversion, but are really important, so here they are!

Claim — Denote by V and W the touchpoints of A-mixtillinear incircle with AB and BC, respectively. Also let the intersections of $\angle BAC$ bisector with BC and (ABC) be E and F, respectively. Finally, let D be the touchpoint of incircle with BC. We have that:

- I is the midpoint of the segment VW
- $FEDT$ is a cyclic quadrilateral



Let us now solve some problems from actual math olympiads!

§4 Problems involving $\sqrt{bc}$ inversion

Problem 4.1 (Some Dumpty point property)**.** Let triangle ABC be given. Let K be the intersection of A-symmedian with (ABC). Also let D_A (the A-Dumpty point of $\triangle ABC$) be the intersection of AK and (BOC). Prove that D_A is the midpoint of segment AK.

Problem 4.2 (Serbian National Olympiad 2013/3)**.** Let M, N and P be midpoints of sides BC, AC and AB, respectively, and let O be circumcenter of acute-angled triangle ABC. Circumcircles of triangles BOC and MNP intersect at two different points X

and Y inside of triangle ABC. Prove that

$$\angle BAX = \angle CAY.$$

Problem 4.3 (IMO Shortlist 2017/G3). Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

Problem 4.4 (USA TST 2016/2). Let ABC be a scalene triangle with circumcircle Ω, and suppose the incircle of ABC touches BC at D. The angle bisector of $\angle A$ meets BC and Ω at E and F. The circumcircle of $\triangle DEF$ intersects the A-excircle at S_1, S_2, and Ω at $T \neq F$. Prove that line AT passes through either S_1 or S_2.

Problem 4.5 (Japan TST 2022/6). Given is a circle Γ with diameter MN and a point A inside Γ. The circle with center N, passing through A, meets Γ at B and C. Let $P, Q \in BC$, such that $\angle BAP = \angle QAC$. The lines NP, NQ meet Γ at X, Y, respectively. Prove that AM, PY, QX are concurrent.

§5 Combinatorial detour

Sometimes inversion might come in handy even while solving combinatorial geometry problems! One very fresh example of that is this year's Baltic Way problem 15, which no team had successfully solved.

Problem (Baltic Way 2024/15). There is a set of $N \geq 3$ points in the plane, such that no three of them are collinear. Three points A, B, C in the set are said to form a Baltic triangle if no other point in the set lies on the circumcircle of triangle ABC. Assume that there exists at least one Baltic triangle. Show that there exist at least $\frac{N}{3}$ Baltic triangles.

Before we solve the problem, we will prove a surprising theorem, which will play a crucial role in our solution:

Theorem (Sylvester-Gallai Theorem)
If $n \geq 3$ points in the plane do not all lie on a single line, then there exists a line containing exactly two of the points.

Now we have all the tools to solve this year's toughest Baltic Way problem!

§6 More inversion problems!

Problem 6.1 (ELMO 2010/6). Let ABC be a triangle with circumcircle ω, incenter I, and A-excenter I_A. Let the incircle and the A-excircle hit BC at D and E, respectively, and let M be the midpoint of arc BC without A. Consider the circle tangent to BC at D and arc BAC at T. If TI intersects ω again at S, prove that SI_A and ME meet on ω.

Problem 6.2 (IMO 1996/2). Let P be a point inside a triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that the lines AP, BD, CE meet at a point.

Problem 6.3 (IMO 2010/2)**.** Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the intersection point of the lines EI and DG lies on Γ.

Problem 6.4 (IMO 2015/3)**.** Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$ and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order.
Prove that the circumcircles of triangles KQH and FKM are tangent to each other.