Estonian Math Competitions
2024/2025

University of Tartu Youth Academy
Tartu 2025

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Estonian Mathematical Olympiad

https://www.math.olympiaadid.ut.ee/

Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds: at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round.

In each round of the Olympiad, separate problem sets are given to the students of each grade from the 7th to the 12th. In the last four years, the final round was organized also to grades 7 and 8; previously, these two grades participated at school and regional levels only. About 25 students of each grade reach the final round. Some towns, regions and schools organize math olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in spring in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

In addition to the Olympiad, open contests take place in September and in December. In addition to students of Estonian middle and secondary schools, Estonian citizens who are studying abroad may also participate in these contests. The participants must have never enrolled in a university or other higher educational institution. The contestants compete in two categories: Juniors and Seniors. In the former category, only students up to the 10th grade may participate. Being successful in the open contests generally assumes knowledge outside the school curriculum.

Based on the results of all competitions during the year, about 20 IMO team candidates are selected. The IMO team selection contest for them is held in April or May in two rounds. Each round is an IMO-style two-day competition with 4.5 hours to solve 3 problems on both days. Some problems in our selection contest are at the level of difficulty of the IMO but easier problems are usually also included.

The problems of previous competitions can be downloaded at the Estonian Mathematical Olympiads website.

Problems Listed by Topic

Number theory: O1, O3, O4, O8, O10, O14, O16, F1, F6, F11, F20, F25, F29
Algebra: O5, O9, O11, O17, F2, F8, F12, F16, F21, F24, F26, S2
Geometry: O2, O6, O12, O18, F3, F5, F7, F9, F13, F17, F19, F22, F27, S3, S4
Discrete mathematics: O7, O13, O15, F4, F10, F14, F15, F18, F23, F28, S1

Problems

Selected Problems from Open Contests

O1 *(Juniors.)* Call a positive integer n *interesting* if both the sum of digits of n and the sum of digits of $n+1$ are perfect squares, whereas n and $n+1$ have the same number of digits. Find all positive integers k for which there exists an interesting k-digit number.

O2 *(Juniors.)* Let $ABCD$ be a rectangle. The bisector of the angle CAD meets the side CD at point L. Let M be the midpoint of the line segment AL. The line DM meets lines AC and AB at points E and F, respectively. Given that line segments AE and AF are equal, prove that $ABCD$ is a square.

O3 *(Juniors.)* A TV show airs every 28 days. This century there was a year when the show aired on both January 1 and January 29. In how many years will the show air twice in January again?

O4 *(Juniors.)* Digits A, B, C are given distinct values from 1 to 9 to make the value of the expression $2024 \cdot AB \cdot CC \cdot BA$ a perfect square. How many distinct values can the expression $A+B+C$ obtain?

O5 *(Juniors.)* Solve the system of equations

$$\begin{cases} x + y = z, \\ x^2 + y^2 = 4z, \\ x^3 + y^3 = 18z. \end{cases}$$

O6 *(Juniors.)* In an isosceles triangle ABC with $AB = AC$, the bisector of the angle BAC intersects BC at D. The bisector of the angle ABC intersects the perpendicular bisector of AD at E. Prove that the bisector of the angle ACB is perpendicular to DE.

O7 *(Juniors.)* The website of a company has multiple pages, which may point to each other. A list containing each of these pages exactly once is called *natural*, if whenever one page points to another, it is written before the other page.

The webmaster of the company has a natural list of pages. Then she reorders the pages, writing down first all pages which are not pointed to by any other page, then all pages which the first page of the original list points to, then all pages which the second page of the original list points to and which are not listed yet etc.

(a) It is known that each page is pointed to by at most one other page. Can we be certain that the new list is natural?

(b) It is not known whether each page is pointed to by at most one other page. Can we be certain that at least one possible new list is natural?

O8 *(Seniors.)* Determine all pairs (m, n) of natural numbers that satisfy $m - n = 96$ and $\operatorname{lcm}(m, n) = 2024$.

O9 *(Seniors.)* According to a message sent by extraterrestrial creatures who are millions of years ahead of us in development, the height of the highest two places of their planet, measured from the sea level, is h, whereas the lowest point on mainland has height l (where $h \geqslant 0 \geqslant l$). The radius of the planet (i.e., the distance of the sea level from the centre of the planet) is r. Express the largest enabled by these conditions distance between two points on this planet, one of which can be visible from the other one.

O10 *(Seniors.)* A positive integer m is called *usual* if the square of every prime divisor of m is less than m.

(a) Prove that there are infinitely many positive integers n such that both n and $n+1$ are usual.

(b) Is there a positive integer n such that n, $n+1$ and $n+2$ are all usual?

O11 *(Seniors.)* Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f\left(y^2 f(x) - f(xy)\right) = f\left(y^2\right) + 2\left(x^2 - f(x)\right)(f(y) - 1) + 1$$

for all real numbers x and y.

O12 *(Seniors.)* Let O be the circumcentre of an acute triangle ABC. Points D and E are chosen on the side BC such that AD is an altitude of the triangle ABC and AE bisects the angle CAD. Bisectors of the triangle AOB meet at point J. Prove that the triangle JBE is isosceles.

O13 *(Seniors.)* A gardener Andres wants to plant one currant bush to each cell of his garden of shape 24×2024. He wants to plant as many blackcurrant bushes as possible under the following conditions: There must be at least one redcurrant and at least one whitecurrant bush, and for any cell with a blackcurrant bush, cells that have a common side with it must contain equally many redcurrant and whitecurrant bushes (maybe also 0 of both). Find the largest number of blackcurrant bushes Andres can plant.

O14 *(Seniors.)* Is there a positive integer n such that 88 divides $2^n + n^3$?

O15 *(Seniors.)* Let $n \geqslant 3$ be an integer. In an $n \times n$ grid there are three invisible monsters: one in the upper right corner square and one in each of its neighboring squares. In a 2×2 area in the opposite corner of the grid some frogs are placed. A square may contain more than one frog.

The frogs and the monsters take turns making the following moves. On the frogs' turn, each frog makes a knight move, jumping either two squares up and one to the right or one square up and two to the right. On the monsters' turn, they may make up to a total of 3 knight moves, jumping either two squares down and one to the left or one square down and two to the left. The frogs start. If a frog and a monster ever land on the same square, the monster will eat the frog.

Find the least number of frogs needed to ensure that at least one frog could make it to a spot where both possible jumps would take it off the grid.

O16 *(Seniors.)* Find all triples (x, y, z) of positive integers, such that

$$3 \cdot x! + 4 \cdot y! = 5 \cdot z!.$$

O17 *(Seniors.)* Mama snail and her child want to visit a neighbour who lives at distance 75 cm. Every hour, they have planned to use 45 minutes to move and 15 minutes to rest. On the n-th hour, they move $\frac{1}{n^2+1}$ metres forward, but instead of resting, the child pulls them backwards by $\frac{1}{n+1}$ of this hour's distance. Will they ever reach the neighbour, and if so, when?

O18 *(Seniors.)* In an acute triangle ABC, the extension of the altitude AD over D intersects the circumcircle at E. The midpoint of CE is F. The circumcircles of ABC and DEF intersect at $G \neq E$. The foot of the altitude drawn from A to FG is P. Prove that $DA = DP$.

Selected Problems from the Final Round of National Olympiad

F1 *(Grade 7.)* The product abc of positive integers a, b, and c is divisible by 3, and the equations $a = \frac{b^2}{2} = \frac{c}{4}$ hold. Find the smallest possible sum of the numbers a, b, and c under these conditions.

F2 *(Grade 7.)* The letters M, A, T, E, I and K are assigned the numbers 1, 2, 3, 4, 5 and 6 in a certain order so that different letters correspond to different numbers. The sum of the numbers corresponding to the letters of the word MATEMAATIK (taking into account repetitions) is 42 and the sum of the numbers corresponding to the letters of the word KEEMIK (taking into account repetitions) is 13. Find the sum of the numbers corresponding to the letters of the word IT.

F3 *(Grade 7.)* The diagonals of the quadrilateral $ABDE$ meet at C. The segments AB and CE are of equal length 8 cm, and the segments AE and CD are also of equal length. The perimeter of the triangle CDE is 35 cm. Given that $\angle BAC = \angle AEC$, find the perimeter of the pentagon $ABCDE$.

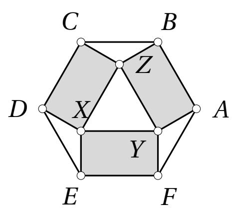
F4 *(Grade 7.)* In how many ways can one choose 5 numbers from the list

$$\frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4, 5, 6, 7, 8$$

so that the product of the chosen numbers is 1?

Remark: Two choices are considered different if one choice contains a number that the other choice does not. The order of numbers is not important.

F5 *(Grade 7.)* Inside a regular hexagon $ABCDEF$, equal rectangles $ABZY$, $CDXZ$, and $EFYX$ are drawn. How much of the area of the hexagon $ABCDEF$ do these rectangles cover?



F6 *(Grade 8.)* Find all natural numbers whose last digit is not zero and deleting the first digit of which gives a number exactly 25 times smaller.

F7 *(Grade 8.)* Let ABC be a triangle and P a point inside it. Let A', B', C' be the reflections of A, B, C from point P, respectively. Find the ratio of the areas of the hexagon $AB'CA'BC'$ and the triangle ABC.

F8 *(Grade 8.)* The price of the old model of smartwatch A differs from the price of the old model of smartwatch B by $p\%$ ($0 < p < 100$). The new model of watch A is $q\%$ more expensive than the old model of watch A, and the new model of watch B is $q\%$ cheaper than the old model of watch B ($0 < q < 100$). The price of the new model of watch A differs from the price of the new model of watch B by $p\%$.

(a) Is the new model of watch B more expensive, cheaper, or just as expensive as the old model of watch A?

(b) Is the new model of watch A more expensive, cheaper, or just as expensive as the old model of watch B?

F9 *(Grade 8.)* The midpoint of the hypotenuse AB of a right-angled triangle ABC is M. A point D lies on the side BC such that the circumcircle of the triangle ACD intersects the line DM at some point K between the points D and M. Let L be the reflection of the point K from the point M. The circumcircles of the triangles ACD and LBC intersect at the point N, $N \neq C$. Find the size of the angle KNL.

F10 *(Grade 8.)* A mother has 7 apples, 6 pears, and 5 oranges. She wants to divide them among 2 children so that each gets the same number of fruits. In how many different ways can this be done?

Remark: We consider the distributions of fruit to be different if a child receives a different number of some types of fruit.

F11 *(Grade 9.)* Call a natural number *twistable* if it does not contain digits 3, 4, 7 and its last digit is not zero. The *twisting* of a twistable number is the number obtained after the following two steps:

- Reverse the order of digits of the given number;
- Twist each digit: 0, 1 and 8 remain unchanged, 2 and 5 are turned into each other, 6 and 9 are turned into each other.

For instance, the twisting of the number 68012 is 51089 and the twisting of the number 69 is 69.

Find all integers that can be represented as the ratio of a twistable positive integer n and its twisting k.

F12 *(Grade 9.)* Prove that the value of the expression

$$x^4 - 2x^3 - 88x^2 + 90x + 2025$$

is positive regardless of the value of x on the real line.

F13 *(Grade 9.)* Let D be a point on the side BC of an acute triangle ABC and let E be a point on the line segment AD. Let F and G be the feet of the altitudes drawn from the vertex D in triangles ABD and ACD, respectively. The line BE intersects the circumcircle of the triangle DEG at point $H \neq E$. Prove that points B, F, G and H are concyclic.

F14 *(Grade 9.)* Ats and Pets take turns to write representations of the number 15 as the sum of three distinct single-digit positive integers. On every move, each player must write a sum that has exactly one common addend with the previous sum, no common addends with the second previous sum, and less than three common addends with any sum written earlier. Ats starts and can choose the first sum arbitrarily. The player who cannot write a sum loses. Which player can win regardless of his opponent's play?

F15 *(Grade 9.)*

(a) Every side and diagonal of a regular 2025-gon is coloured either red or blue. Can it happen that the same number of red and blue line segments meet at each vertex?

(b) The same question if only the diagonals are coloured.

F16 *(Grade 10.)* Anna, Berta and Carol make fruit drinks from syrup. Anna makes a litres of drink by mixing water and syrup in the proportion of $a : 1$. Berta makes b litres of drink by mixing water and syrup in the proportion of $b : 1$. Carol makes c litres of drink by mixing water and syrup in the proportion of $c : 1$. (It is not known if a, b and c are integers.) They make 6 litres of drink in total. Prove that they use at most 2 litres of syrup.

F17 *(Grade 10.)* The incentre of a triangle ABC is I. Points D and E on the sides AB and AC, respectively, satisfy $DI \perp BI$ and $EI \perp CI$. Prove that the line DE is tangent to the incircle of the triangle ABC.

F18 *(Grade 10.)* Let n be any natural number. Find the least natural number k such that it is possible to write a natural number from 1 through k into every cell of an $n \times n$ table in such a way that the sum of every two cells with a common side differs from all other such sums. The numbers in different cells do not have to be distinct.

F19 *(Grade 10.)* A point E is chosen on the side AB of a rectangle $ABCD$ ($E \neq A$, $E \neq B$). The line segments BD and CE intersect at point F. Among the triangles ADE, DEF, DCF, BCF and BEF, there are exactly two pairs of triangles with equal area (the order of components in a pair is not taken into account). Find the ratio of the lengths of the line segments EB and AB.

F20 *(Grade 11.)* Find the least positive integer n such that:

(a) both n and $n + 1$ are divisible by the squares of two distinct prime numbers;

(b) both n and $n + 3$ are divisible by the squares of two distinct prime numbers.

F21 *(Grade 11.)* The coefficients of the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

are real numbers such that $a_i = a_{n-i}$ for every $i = 0, 1, \ldots, n$, and $a_n \neq 0$. Let $x_1, x_2, \ldots, x_k$ be all the real roots of the polynomial $P(x)$ without repetitions.

(a) Prove that

$$|x_1| + |x_2| + \ldots + |x_k| \geqslant k.$$

(b) Is the inequality definitely strict in the case $k > 1$?

F22 *(Grade 11.)* Circles ω_1 and ω_2 touch at point K. The line through the centres of the circles intersects the circle ω_1 once more at point A. A line through the point A intersects the circle ω_1 once more at point B and the circle ω_2 at points C and D, where the points A, B, C, D lie on the line in the order. Given that the line segments AB, BC and CD have equal lengths, find the ratio of the radii of the circles ω_1 and ω_2.

F23 *(Grade 11.)* Find all positive integers n for which all 10^n non-negative integers consisting of n digits can be ordered in such a way that all the following conditions are met:

(1) the first number consists of zeros only;

(2) every two numbers that are consecutive in this order differ at exactly one position and the digits at this position differ by exactly 1;

(3) the last number consists of nines only.

Remark: In this problem, we allow numbers to begin with zero(s).

F24 *(Grade 11.)* Every night, Juku listens to exactly 14 songs from a playlist containing exactly 100 songs. Every time a song ends, the next song is chosen from among all 100 songs with equal probability (the same song may also repeat). Prove that, on more than half of all nights, Juku listens some song more than once.

F25 *(Grade 12.)* Integers are assigned to variables x, y and z to satisfy the equation

$$x^3 + y^3 + z^3 - 3xyz = 2025.$$

Find all possible values of the sum $x + y + z$.

F26 *(Grade 12.)* Find all functions f from the set of all non-negative real numbers to the set of all real numbers such that $f(1) = 1$ and

$$(f(x+y))^2 \leqslant f\left(x^2 - 2xy + y^2\right)$$

for all real numbers x and y that satisfy the inequality $x + y \geqslant 0$.

F27 *(Grade 12.)* In an acute triangle ABC with $AB < AC$, points D, E and F are the feet of the altitudes drawn from vertices A, B and C, respectively. Let the orthocenter of ABC be H and the midpoint of the side BC be M. Point K on the prolongation of the line segment EM beyond M and point L on the line segment FM satisfy $MK = ML = MD$. Prove that points K, L and H lie on a line.

F28 *(Grade 12.)* On a plane, 5 points are chosen arbitrarily. Find the largest possible number of distinct right triangles with all vertices in the chosen points.

F29 *(Grade 12.)* Does there exist a geometric progression, among the members of which there are

(a) 3, 45 and 2025;

(b) 3, $\frac{45}{\sqrt[3]{5}}$ and 2025?

Selected Problems from the IMO Team Selection Contests

S1 Let $n \geqslant 3$ be any natural number. A real number is written into every vertex of a regular n-gon in such a way that numbers in any two neighbouring vertices differ by at most 1. Find the least non-negative real number C such that, regardless of the choice of the numbers in the vertices, there exist two neighbouring vertices in which numbers differ by at most C.

S2 Kati writes the numbers

$$2^0, 2^1, 2^2, \ldots, 2^{100}, 3^0, 3^1, 3^2, \ldots, 3^{100}, 6^0, 6^1, 6^2, \ldots, 6^{100}$$

on the board. In each step, she performs one of the following operations:

(1) She can pick two numbers and replace them with their greatest common divisor and least common multiple; or

(2) She can pick two numbers, one of which is divisible by the other, and replace them with some two numbers whose greatest common divisor and least common multiple would be the two picked numbers.

Find the least and the greatest sum of all numbers on the board that can be achieved via finitely many steps.

S3 The angle bisectors of an acute triangle ABC meet at point I. The line AI meets the circumcircle of the triangle ABC at point D ($D \neq A$) and the side BC at point E. The line BI meets the circumcircle of the triangle CDI at point K whereas the line CI meets the circumcircle of the triangle BDI at point L ($K \neq I$, $L \neq I$).

(a) Prove that the line DI is tangent to the circumcircle of the triangle IKL.

(b) Prove that points A, K, L, E are concyclic.

S4 A triangle ABC with $AB < BC$ and an obtuse angle at vertex B is given. The incircle of the triangle ABC with incentre I touches the sides BC, CA and AB at points D, E and F, respectively. The line AI intersects the side BC at point K. The ray IB intersects the circumcircle of the triangle KIF at point P ($P \neq I$), whereas the ray IC intersects the circumcircle of the triangle KIE at point Q ($Q \neq I$). Prove that the line PQ bisects the line segment DK.

Problems with Solutions

Selected Problems from Open Contests

O1 *(Juniors.)* Call a positive integer n *interesting* if both the sum of digits of n and the sum of digits of $n+1$ are perfect squares, whereas n and $n+1$ have the same number of digits. Find all positive integers k for which there exists an interesting k-digit number.

Answer: All positive integers $k \geqslant 10$.

Solution: For any positive integer a, let $s(a)$ denote the sum of digits of a. Clearly $s(n+1) = s(n)+1$ unless the last digit of $n+1$ is zero. The only two consecutive integers that are both perfect squares are 0 and 1, but $s(n) = 0$ is impossible for a positive n. The contradiction shows that the last digit of $n+1$ must be 0 and the last digit of n must be 9.

Let n be an interesting positive integer that ends with exactly m digits 9. Clearly $s(n) - s(n+1) = 9m - 1$. Note that a perfect square is congruent to 0, 1, 4 or 7 modulo 9. As $s(n) - s(n+1) = 9m - 1 \equiv 8 \pmod 9$, we must have $s(n) \equiv 0 \pmod 9$ and $s(n+1) \equiv 1 \pmod 9$. But $s(n+1) = 1$ is impossible, because this would require $n+1$ to be a power of 10, in the case of which n and $n+1$ would not consist of the same number of digits. Thus $s(n+1) \geqslant 8^2 = 64$, as 8^2 is the next smallest perfect square congruent to 1 modulo 9. But as $s(n) > s(n+1)$, we have $s(n) \geqslant 9^2 = 81$. This implies that n must contain at least 9 digits. Containing exactly 9 digits would be possible only if n consisted entirely of nines, but in such case $n+1$ would contain one more digit. Hence n must contain at least 10 digits.

Consider now the 10-digit number $n = 7888888899$. In this case $m = 2$, $s(n) = 81 = 9^2$ and $s(n+1) = 81 - 9 \cdot 2 + 1 = 64 = 8^2$, thus n is interesting. Inserting zeros between 7 and 8 does not influence $s(n)$ and $s(n+1)$. So we can obtain an m-digit interesting number for every $m \geqslant 10$.

Remark: Considering congruences modulo 9 is unnecessary. Instead, one could just calculate all differences of perfect squares $2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2$ and check by brute force that none of them can be represented in the form $9m - 1$.

O2 *(Juniors.)* Let $ABCD$ be a rectangle. The bisector of the angle CAD meets the side CD at point L. Let M be the midpoint of the line segment AL. The line DM meets lines AC and AB at points E and F, respectively. Given that line segments AE and AF are equal, prove that $ABCD$ is a square.

Solution 1: Let $\angle DAL = \angle LAC = \alpha$. Then $\angle FAE = 90^\circ - 2\alpha$ (Fig. 1). As $AE = AF$, we obtain $\angle AFE = \frac{180^\circ - (90^\circ - 2\alpha)}{2} = 45^\circ + \alpha$. But as M bisects the hypotenuse AL of the right triangle ALD, it follows that M is the circumcentre of ALD, implying $MA = MD$. Hence $\angle MDA = \angle MAD = \alpha$, implying $\angle AFE = \angle AFD = 90^\circ - \alpha$. We obtain $45^\circ + \alpha = 90^\circ - \alpha$ which implies $\angle CAD = 2\alpha = 45^\circ$. Hence also $\angle ACD = 45^\circ$, implying $AD = CD$. Consequently, $ABCD$ is a square.

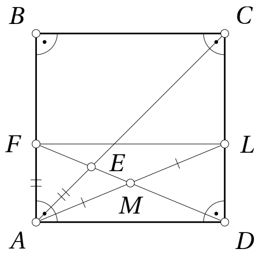


Fig. 1

Solution 2: As $AE = AF$, the triangle AEF is isosceles with vertex angle at A (Fig. 1). Denote $\angle AEF = \angle AFE = \beta$.
Note that triangles MAF and MLD are equal because $\angle FMA = \angle DML$, $\angle MAF = \angle MLD$ and $MA = ML$. Thus $AF = DL$ which shows that $ADLF$ is a rectangle. As the diagonals of a rectangle are equal and bisect each other, we have $MA = MF$, implying that the triangle MAF is isosceles with vertex angle at M. Hence $\angle MAF = \angle MFA = \beta$.
As $\angle EAF = 180° - 2\beta$, we have $\angle MAE = \beta - (180° - 2\beta) = 3\beta - 180°$. Hence also $\angle MAD = 3\beta - 180°$, because AM bisects the angle DAE. We obtain the equation $\beta + (3\beta - 180°) = 90°$ which gives $\beta = 67.5°$. Thus $\angle BAC = 180° - 2 \cdot 67.5° = 45°$, implying that the diagnonal of the rectangle $ABCD$ bisects its angle. Consequently, $ABCD$ is a square.

O3 *(Juniors.)* A TV show airs every 28 days. This century there was a year when the show aired on both January 1 and January 29. In how many years will the show air twice in January again?
Answer: 21.
Solution: In a common year, there are $365 = 13 \cdot 28 + 1$ days. Thus, after a common year, the dates of the airings shift 1 day earlier. After a leap year, the airings shift 2 days earlier. In every 4 years, there are 3 common years and 1 leap year, which means a total shift of $3 \cdot 1 + 2 = 5$ days. Thus in $5 \cdot 4 = 20$ years the total shift is $5 \cdot 5 = 25$ days.
Thus, if in year a the show airs on January 29, then in year $a + 20$ it will air on January 4, and in the intermediate years it will air between these dates. In these years, there cannot be a second airing in January. However, in year $a + 21$, the first airing will be on January 2 or January 3, which also means a second airing in January. Thus 21 years is the desired answer.
Remark. Years divisible by 100 (but not by 400) are not leap years due to the correction of the Gregorian calendar. This could affect the answer of the problem. To avoid this, the condition ”This century there was a year” is necessary.

O4 *(Juniors.)* Digits A, B, C are given distinct values from 1 to 9 to make the value of the expression $2024 \cdot AB \cdot CC \cdot BA$ a perfect square. How many distinct values can the expression $A + B + C$ obtain?

Answer: 6.
Solution: Denote the value of the given expression by k. We know that $2024 = 2^3 \cdot 11 \cdot 23$ and $CC = C \cdot 11$. In a perfect square, all prime exponents are even. Thus one of the numbers AB, BA, CC must be divisible by 23. This cannot be CC, so let it be AB without loss of generality (the other option is symmetrical). The two-digit multiples of 23 are 23, 46, 69 and 92.

- If $AB = 23$, then $BA = 32 = 2^5$, so $k = 2^8 \cdot 11^2 \cdot 23^2 \cdot C$. Then k is a perfect square iff C is a perfect square, meaning $C = 1, C = 4$ or $C = 9$. The sum $A + B + C$ is 6, 9 or 14 respectively.
- If $AB = 46$, then $BA = 64 = 2^6$, so $k = 2^{10} \cdot 11^2 \cdot 23^2 \cdot C$. Then k is a perfect square iff C is a perfect square, meaning $C = 1$ or $C = 9$ (the digit 4 is already in use). The sum $A + B + C$ is 11 or 19 respectively.
- If $AB = 69$, then $BA = 96 = 2^5 \cdot 3$, so $k = 2^8 \cdot 3^2 \cdot 11^2 \cdot 23^2 \cdot C$. Then k is a perfect square iff C is a perfect square, meaning $C = 1$ or $C = 4$ (the digit 9 is already in use). The sum $A + B + C$ is 16 or 19 respectively.
- If $AB = 92$, then $BA = 29$, so $k = 2^5 \cdot 11^2 \cdot 23^2 \cdot 29 \cdot C$. This cannot be a perfect square, as C cannot be divisible by 29.

The sum $A + B + C$ can thus obtain 6 different values (19 was present in two different cases).

O5 *(Juniors.)* Solve the system of equations

$$\begin{cases} x + y = z, \\ x^2 + y^2 = 4z, \\ x^3 + y^3 = 18z. \end{cases}$$

Answer: $x = 0, y = 0, z = 0$ or $x = 3 + \sqrt{3}, y = 3 - \sqrt{3}, z = 6$ or $x = 3 - \sqrt{3}$, $y = 3 + \sqrt{3}, z = 6$.
Solution 1: If $x \neq 0$ or $y \neq 0$, then from the second equation $z > 0$. Thus $z = 0$ can only hold if $x = y = 0$. The triple $(x, y, z) = (0, 0, 0)$ satisfies all equations. Now assume $z \neq 0$.
Squaring the first equation yields $x^2 + 2xy + y^2 = z^2$. Subtracting the second equation yields

$$2xy = z^2 - 4z. \tag{1}$$

Cubing the first equation yields $x^3 + 3x^2y + 3xy^2 + y^3 = z^3$. Subtracting the third equation and factoring yields

$$3xy(x + y) = z\left(z^2 - 18\right).$$

Using $x + y = z \neq 0$, this can be reduced to

$$3xy = z^2 - 18. \tag{2}$$

Expressing xy from both (1) and (2) yields $3\left(z^2 - 4z\right) = 2\left(z^2 - 18\right)$. This simplifies to $z^2 - 12z + 36 = 0$ or $(z - 6)^2 = 0$, from which $z = 6$.

Now equation (1) yields $2xy = 36 - 24 = 12$, from which $xy = 6$. On the other hand $x + y = z = 6$. Combining by Viète's formulas yields the quadratic equation $x^2 - 6x + 6 = 0$, from which $x = 3 \pm \sqrt{3}$ and respectively $y = 3 \mp \sqrt{3}$. The triples $(x, y, z) = (3 + \sqrt{3}, 3 - \sqrt{3}, 6)$ and $(x, y, z) = (3 - \sqrt{3}, 3 + \sqrt{3}, 6)$ satisfy all three equations.

Solution 2: Like in Solution 1, we show that $z = 0$ yields only the solution $(x, y, z) = (0, 0, 0)$. We also similarly deduce (1). We then use the identity $x^3 + y^3 = (x + y)\left(x^2 - xy + y^2\right)$. Substituting $x + y$, $x^2 + y^2$ and $x^3 + y^3$ from the given equations and xy from (1) yields

$$18z = z\left(4z - \frac{z^2 - 4z}{2}\right).$$

Dividing both sides by $z \neq 0$ and simplifying yields $z^2 - 12z + 36 = 0$ or $(z - 6)^2 = 0$, from which $z = 6$. We proceed like in Solution 1.

O6 *(Juniors.)* In an isosceles triangle ABC with $AB = AC$, the bisector of the angle BAC intersects BC at D. The bisector of the angle ABC intersects the perpendicular bisector of AD at E. Prove that the bisector of the angle ACB is perpendicular to DE.

Solution 1: As the A-bisector is also the altitude, we have $BD \perp AD$. Let I be the incenter of ABC and $E' \neq B$ the intersection of BI and the circumcircle of ABD (Fig. 2). As the angles $\angle ABE' = \angle DBE'$ correspond to the arcs $E'A$ and $E'D$ of this circle, we have $E'A = E'D$. Thus E' lies on the perpendicular bisector of AD, meaning $E' = E$. Hence $ABDE$ is a cyclic quadrilateral (Fig. 3).

It remains to verify that $\angle EDI + \angle CID = 90^\circ$, which would yield $CI \perp DE$, as desired. To show this, let $\angle ACB = \angle ABC = \gamma$. Then

$$\angle EDI = \angle EDA = \angle EBA = \frac{\gamma}{2},$$

whereas

$$\angle CID = 180^\circ - \angle CDI - \angle DCI = 180^\circ - 90^\circ - \frac{\gamma}{2} = 90^\circ - \frac{\gamma}{2}.$$

Thus $\angle EDI + \angle CID = 90^\circ$, finishing the proof.

Solution 2: As the A-bisector is also the altitude, we have $BD \perp AD$. Thus the perpendicular bisector of AD is parallel to BC, therefore containing the

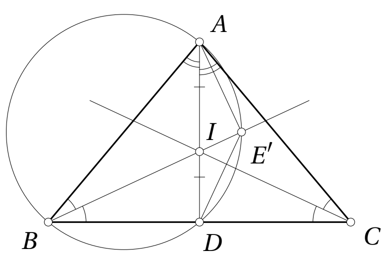


Fig. 2

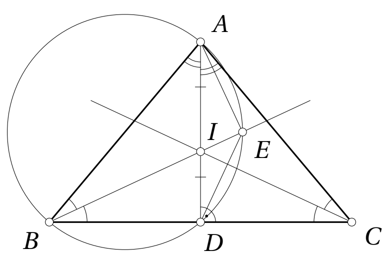


Fig. 3

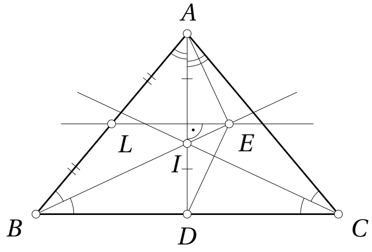


Fig. 4

midline of ABC and bisecting AB and AC. Let L be the midpoint of AB and I the incenter of ABC (Fig. 4). Then $\angle LEB = \angle CBE = \angle LBE$, meaning that LBE is isosceles and $LE = LB = LA$. Thus L is the circumcenter of ABE and AB is a diameter of the circle. Now as $ADB = 90°$, the point D also lies on this circle, meaning $ABDE$ is cyclic. We proceed like in Solution 1.

Solution 3: As the A-bisector is also the altitude, we have $BD \perp AD$. Thus the perpendicular bisector of AD is parallel to BC, therefore containing the midline of ABC and bisecting AB and AC. Let L amd M be the midpoints of AB and AC, respectively. Then $\angle LEB = \angle CBE = \angle LBE$, meaning that LBE is isosceles and $LE = LB = LA$. Also $MC = MA = LB = LE$, and as LM is the midline of ABC, we have $LM = \frac{1}{2}BC = CD$.

Let N be the intersection of DE and AC; depending on the orientation of points (Figures 5 and 6) we have

$$CN = CM \pm MN,$$
$$CD = LM = LE \pm EM = CM \pm ME.$$

Triangles NCD and NME are similar by parallel sides, giving $\frac{CD}{ME} = \frac{CN}{MN}$. Substituting in the previous equation, we get $\frac{CM \pm ME}{ME} = \frac{CM \pm MN}{MN}$, from

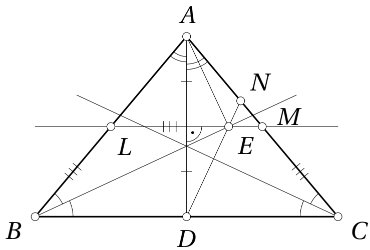


Fig. 5

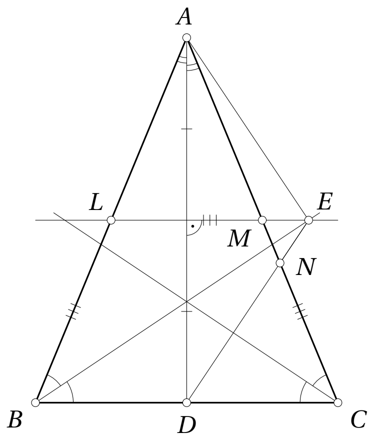


Fig. 6

which $\frac{CM}{ME} \pm 1 = \frac{CM}{MN} \pm 1$, which in turn implies $\frac{CM}{ME} = \frac{CM}{MN}$. Therefore $ME = MN$, from which $CD = CN$. The claim of the problem follows, as the C-bisector in the isosceles triangle CDN and the side DN are perpendicular.

Solution 4: Let I be the incenter of ABC. Clearly D is the point of tangency of the incircle and side BC. Let N be the tangency point with side AC (Fig. 7), then $CD = CN$. In the isosceles triangle CDN, the line CI is both the angle bisector and altitude, so DN and CI are perpendicular. It remains to show that E lies on DN.

By the Iran lemma, the angle bisector BI and the extensions of the incircle chord DN and the midline corresponding to BC intersect at one point. Of those, the first and the third intersect at E, so E must lie on DN, as desired.

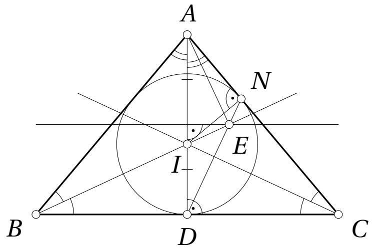


Fig. 7

O7 *(Juniors.)* The website of a company has multiple pages, which may point to each other. A list containing each of these pages exactly once is called *natural*, if whenever one page points to another, it is written before the other page.

The webmaster of the company has a natural list of pages. Then she re-orders the pages, writing down first all pages which are not pointed to by any other page, then all pages which the first page of the original list points to, then all pages which the second page of the original list points to and which are not listed yet etc.

(a) It is known that each page is pointed to by at most one other page. Can we be certain that the new list is natural?

(b) It is not known whether each page is pointed to by at most one other page. Can we be certain that at least one possible new list is natural?

Answer: (a) Yes; (b) No.

Solution: We write $x \to y$ whenever the page x points to the page y.

(a) Consider any two pages a and b with $a \to b$. By the assumption, b is not pointed to by any other page. Thus b is located in the new list in the group of pages pointed to by page a. We consider two cases.

- If a is not pointed to by any page, then it is in the first group, and thus also before b.
- If a is pointed to by a page c, then a is in the group of pages pointed to

by c. But as the initial list is natural, c must be before a in it. Thus in the new list, a must be before b.

In either case, a is before b in the new list, making it natural, as desired.

(b) Let there be 4 pages a, b, c, d with $a \to b$, $b \to c$, $c \to d$ and $a \to d$ (Fig. 8). Then the list a, b, c, d is natural. The new list can then be either a, b, d, c or a, d, b, c. But neither of them is natural, as the last page c points to the page d before it.

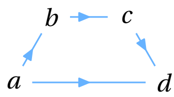


Fig. 8

O8 *(Seniors.)* Determine all pairs (m, n) of natural numbers that satisfy $m - n = 96$ and $\operatorname{lcm}(m, n) = 2024$.

Answer: $(184, 88)$.

Solution 1: As $\operatorname{lcm}(m, n) = 2024 = 8 \cdot 253$ and $8 = 2^3$, at least one of the numbers m and n is divisible by 8. Since $8 \mid 96 = m - n$, the other one must also be divisible by 8. Both m and n are divisors of 2024. All divisors of 2024 that are divisible by 8 are 8, 88, 184 and 2024. The only two of these with difference 96 are 184 and 88. A straightforward check shows that $\operatorname{lcm}(184, 88) = \operatorname{lcm}(8 \cdot 23, 8 \cdot 11) = 8 \cdot 23 \cdot 11 = 2024$ indeed.

Solution 2: As $\operatorname{lcm}(m, n) = 2024$, both m and n are divisors of 2024. All divisors of 2024 are

$$1, 2, 4, 8, 11, 22, 23, 44, 46, 88, 92, 184, 253, 506, 1012, 2024.$$

As $m - n = 96$, we must have $m > 96$. This observation cuts out all cases except $m = 184$, $m = 253$, $m = 506$, $m = 1012$ and $m = 2024$. As differences between 253, 506, 1012 and 2024 are larger than 96, it suffices to check the options $m = 184$ and $m = 253$ which imply $n = 88$ and $n = 157$, respectively. As $157 \nmid 2024$, the only solution can be $(m, n) = (184, 88)$. An easy check shows that $\operatorname{lcm}(184, 88) = \operatorname{lcm}(8 \cdot 23, 8 \cdot 11) = 8 \cdot 23 \cdot 11 = 2024$ indeed.

O9 *(Seniors.)* According to a message sent by extraterrestrial creatures who are millions of years ahead of us in development, the height of the highest two places of their planet, measured from the sea level, is h, whereas the lowest point on mainland has height l (where $h \geqslant 0 \geqslant l$). The radius of the planet (i.e., the distance of the sea level from the centre of the planet) is r. Express the largest enabled by these conditions distance between two points on this planet, one of which can be visible from the other one.

Answer: $2\sqrt{(r+h)^2 - (r+l)^2}$, or equivalently, $2\sqrt{(2r+h+l)(h-l)}$.

Solution: For the distance between two points visible from each other to be

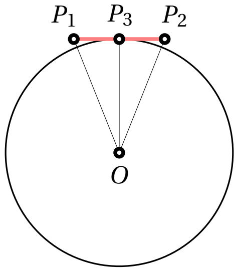


Fig. 9

maximal, the line connecting these points must be a tangent of the planet, otherwise one could increase the distance by pushing the points along the surface of the planet farther away. Let the centre be O; let the two points under consideration be P_1 and P_2 with the tangent point P_3 between them (Fig. 9). Denote $OP_i = d_i$ for $i = 1, 2, 3$. As the tangent line is perpendicular to the radius drawn to the tangent point, OP_1P_3 and OP_2P_3 are right triangles with hypothenuses OP_1 and OP_2, respectively. Hence

$$P_1P_2 = P_1P_3 + P_2P_3 = \sqrt{d_1^2 - d_3^2} + \sqrt{d_2^2 - d_3^2}.$$

The value of this expression is maximal if d_1 and d_2 are as large as possible and d_3 is as small as possible, i.e., $d_1 = d_2 = r + h$ and $d_3 = r + l$. Substituting these values gives the desired distance $2\sqrt{(r+h)^2 - (r+l)^2}$, or equivalently, $2\sqrt{(2r+h+l)(h-l)}$.

O10 *(Seniors.)* A positive integer m is called *usual* if the square of every prime divisor of m is less than m.

(a) Prove that there are infinitely many positive integers n such that both n and $n+1$ are usual.

(b) Is there a positive integer n such that n, $n+1$ and $n+2$ are all usual?

Answer: (b) Yes.

Solution 1: Let a be a composite number such that $a+1$ is composite, too. Then a^2 is usual since each of its prime divisors is a prime divisor of a and therefore less than a. We show that a^2-1 is usual, too. To this end, observe that $a^2 - 1 = (a-1)(a+1)$. As $a+1$ is composite, all prime divisors of $a+1$ are less than $a+1$. But as the difference of $a+1$ and its arbitrary prime divisor is also divisible by this divisor, the prime divisors of $a+1$ cannot be larger than $a-1$. Thus no prime divisor of of a^2-1 is larger than $a-1$, meaning that the squares of prime divisors of a^2-1 do not exceed $(a-1)^2$. As $\left(a^2-1\right) - (a-1)^2 = (a-1)\left((a+1)-(a-1)\right) = 2(a-1) > 0$, squares of prime divisors of a^2-1 are less than a^2-1, implying that a^2-1 is usual. Consequently, $n = a^2-1$ is a suitable example.

As there exist arbitrarily long sequences of consecutive composite numbers, one can choose two consecutive composite numbers in infinitely many ways. Hence there exist infinitely many integers with the desired property. It remains to notice that $a = 21$ implies $a^2 + 1 = 442 = 2 \cdot 13 \cdot 17$, whereas $2^2 < 13^2 < 17^2 = 289 < 442$. Hence $(440, 441, 442)$ is a triple of consecutive positive integers, all of which are usual.

Solution 2: Consider the so-called Pell's equation $x^2 - 2y^2 = 1$. It has the trivial solution $(x_0, y_0) = (1, 0)$, and whenever (x_{k-1}, y_{k-1}) is a solution, $(x_k, y_k) = (3x_{k-1} + 4y_{k-1}, 2x_{k-1} + 3y_{k-1})$ is a solution, too.

As $x_0 < x_1 < x_2 < \ldots$ and $y_0 < y_1 < y_2 < \ldots$, all these solutions are distinct. Note that $3 \mid y_0$ and, whenever $3 \mid y_{2k}$, also $3 \mid 3x_{2k} + 4y_{2k} = x_{2k+1}$, and whenever $3 \mid x_{2k+1}$, also $3 \mid 2x_{2k+1} + 3y_{2k+1} = y_{2k+2}$. Hence $3 \mid x_{2k+1}$ for every natural number k.

Let k be any positive integer. We show that $n = 2y_{2k+1}^2$ meets the conditions. To this end, let p be any prime divisor of n. If $p > 2$ then $p \mid y_{2k+1}^2$, implying $p \mid y_{2k+1}$ which gives $p \leqslant y_{2k+1}$. But if $p = 2$ then $p = y_1 < y_{2k+1}$ still. Thus $p^2 \leqslant y_{2k+1}^2 < 2y_{2k+1}^2 = n$ in any case. Let p now be a prime divisor of $n + 1 = x_{2k+1}^2$. Then also $p \mid x_{2k+1}$. We previously showed that $3 \mid x_{2k+1}$. As $3 = x_1 < x_{2k+1}$, the number x_{2k+1} is composite, implying that $p < x_{2k+1}$. Thus $p^2 < x_{2k+1}^2 = n + 1$. Consequently, n and $n + 1$ are both usual. This solves part (a) of the problem since the parameter k can be chosen in infinitely many ways which all produce different solutions.

To solve part (b), it suffices to note that if $k = 1$ then $n = 2 \cdot 70^2 = 9800$, $n + 1 = 99^2 = 9801$ and $n + 2 = 9802 = 2 \cdot 13^2 \cdot 29$. Moreover, observe that $2^2 < 13^2 < 29^2 = 841 < 9802$. Hence 9800, 9801 and 9802 are three consecutive positive integers which are all usual.

Remark 1: In Solution 2, one can consider Pell's equation $x^2 - dy^2 = 1$ for any natural number d which is not a perfect square.

Remark 2: The least triple of consecutive positive integers, all of which are usual, is $(350, 351, 352)$. To find the triple $(440, 441, 442)$ in Solution 1, one could note that $4^2 + 1 = 17$ and thus all numbers in the form $(17k + 4)^2 + 1$ are divisible by 17. Taking $k = 1$ leads to the desired triple. Similarly, we can start from the equality $(-2)^2 + 1 = 5$, but then we have to study more cases. The first suitable triple would be $(38^2 - 1, 38^2, 38^2 + 1)$.

O11 *(Seniors.)* Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f\left(y^2 f(x) - f(xy)\right) = f\left(y^2\right) + 2\left(x^2 - f(x)\right)(f(y) - 1) + 1$$

for all real numbers x and y.

Answer: $f(x) = x^2 + 1$.

Solution 1: Substituting $y = 1$ into the given equation, we obtain

$$f(0) = f(1) + 2\left(x^2 - f(x)\right)(f(1) - 1) + 1$$

which is equivalent to

$$2f(x)(f(1) - 1) = f(1) - f(0) + 1 + 2x^2(f(1) - 1). \qquad (3)$$

Hence either $f(1) = 1$ or $f(x) = x^2 + \frac{f(1)-f(0)+1}{2(f(1)-1)}$ for every real number x.
If $f(1) = 1$ then (3) implies $0 = 1 - f(0) + 1$, or equivalently, $f(0) = 2$. Substituting now $y = 0$ into the original equation and applying $f(0) = 2$ gives us

$$f(-2) = 2 + 2\left(x^2 - f(x)\right) + 1,$$

or equivalently, $f(x) = x^2 + c$ where $c = \frac{3-f(-2)}{2}$. In the other case, $f(x)$ is expressed in the same form with $c = \frac{f(1)-f(0)+1}{2(f(1)-1)}$.
We show that the function $f(x) = x^2 + c$ satisfies the original equation if and only if $c = 1$. Substituting $f(x) = x^2 + c$ into the original equation and simplifying leads to

$$\left(c^2 - 1\right) y^4 + 2c(1-c)y^2 + \left(3c^2 - 2c - 1\right) = 0.$$

This equality must hold for every real number y. For that, all coefficients in the left hand side must be zeros. From the leading term, we get $c^2 - 1 = 0$, implying $c = 1$ or $c = -1$. From the quadratic term, we get $2c(1-c) = 0$, implying $c = 0$ or $c = 1$. Altogether, only $c = 1$ works. It makes the constant term also zero. Hence $f(x) = x^2 + 1$ is the only function that satisfies the given equation.
Solution 2: Substituting $y = 0$ into the original equation gives

$$f(-f(0)) = f(0) + 2\left(x^2 - f(x)\right)(f(0) - 1) + 1,$$

or equivalently,

$$2f(x)(f(0) - 1) = f(0) - f(-f(0)) + 1 + 2x^2(f(0) - 1). \qquad (4)$$

This implies that either $f(0) = 1$ or $f(x) = x^2 + \frac{f(0)-f(-f(0))+1}{2(f(0)-1)}$ for any real number x.
Consider the case $f(0) = 1$. Substituting $x = 0$ into the original equation leads to

$$f\left(y^2 - 1\right) = f\left(y^2\right) - 2(f(y) - 1) + 1,$$

which must hold for any real number y. Substituting $-y$ for y gives

$$f\left(y^2 - 1\right) = f\left(y^2\right) - 2(f(-y) - 1) + 1,$$

which must also hold for any real number y. Hence $f(y) = f(-y)$ for any real number y, i.e., f is an even function. Eliminating the two terms with $f(0) - 1$ in (4) gives $0 = 1 - f(-1) + 1$, or equivalently, $f(-1) = 2$. Substituting now $y = -1$ into the original equation gives

$$f(f(x) - f(-x)) = f(1) + 2\left(x^2 - f(x)\right)(f(-1) - 1) + 1. \qquad (5)$$

As f is an even function and $f(1) = f(-1) = 2$, the equation (5) simplifies to $1 = 2 + 2\left(x^2 - f(x)\right) + 1$ which is equivalent to $f(x) = x^2 + 1$.

Hence $f(x) = x^2 + c$ for any real number x, where c is some constant. We proceed like in Solution 1.

O12 *(Seniors.)* Let O be the circumcentre of an acute triangle ABC. Points D and E are chosen on the side BC such that AD is an altitude of the triangle ABC and AE bisects the angle CAD. Bisectors of the triangle AOB meet at point J. Prove that the triangle JBE is isosceles.

Solution 1: Let $\angle BCA = \gamma$ and let M be the point of intersection of lines OJ and AB (Figures 10 and 11). As $OA = OB$ and OM bisects the angle AOB, we have $OM \perp AB$. We also have $\angle AOJ = \frac{1}{2}\angle AOB = \angle ACB = \gamma$, because O lies inside the triangle ABC. As $\angle AMO = 90^\circ = \angle ADC$, triangles AMO and ADC are similar. Hence also $\angle MAO = \angle DAC = 90^\circ - \gamma$, implying

$$\angle MAJ = \angle JAO = \angle DAE = \angle EAC = \frac{1}{2}\left(90^\circ - \gamma\right). \tag{6}$$

Hence triangles AJO and AEC are similar, too. This implies the similarity of triangles AJE and AOC (spiral similarity). As $OA = OC$, the latter similarity implies $JA = JE$. By symmetry in the isosceles triangle AOB, we obtain $JA = JB$. Consequently, $JB = JE$. Hence the triangle JBE is isosceles.

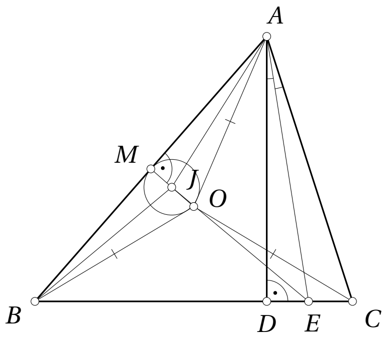


Fig. 10

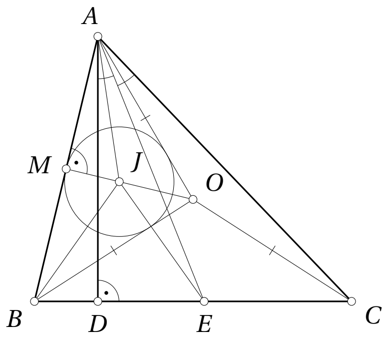


Fig. 11

Solution 2: Let $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. We start by proving the equalities (6) as in Solution 1. Next, let lines OJ and BC meet in F, whereas let lines AJ and BC meet in X (Figures 12 and 13 depict situations that differ by the order of points E and F). We show that points A, J, E, F are concyclic. From equalities (6) we get

$$\begin{aligned}\angle JAE &= \angle BAC - \angle BAJ - \angle EAC = \alpha - \tfrac{1}{2}\left(90^\circ - \gamma\right) - \tfrac{1}{2}\left(90^\circ - \gamma\right)\\ &= \alpha - \left(90^\circ - \gamma\right) = \alpha + \gamma - 90^\circ = 180^\circ - \beta - 90^\circ = 90^\circ - \beta.\end{aligned}$$

As $OA = OB$, the line OJ is the perpendicular bisector of the line segment AB. Thus

$$\angle XFJ = \angle BFJ = 90^\circ - \angle ABF = 90^\circ - \beta.$$

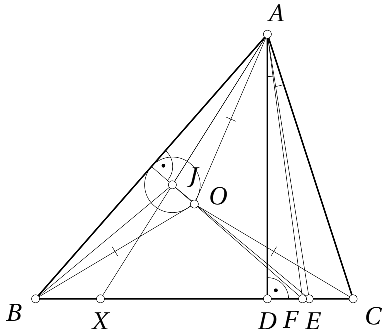


Fig. 12

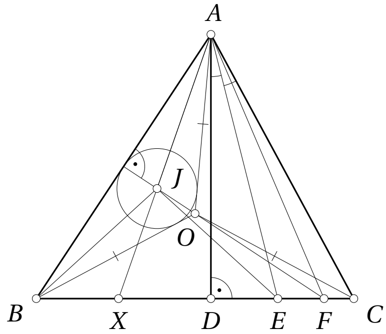


Fig. 13

Altogether, we have $\angle XAE = \angle JAE = 90^\circ - \beta = \angle XFJ$, implying that A, J, E, F are concyclic.
Therefore $\angle JEA = \angle JFA = \angle JFB = \angle JFX = \angle XAE = \angle JAE$, implying $JA = JE$. But $JA = JB$ as J lies on the perpendicular bisector of the line segment AB. Consequently, $JB = JE$. Hence the triangle JBE is isosceles.
Solution 3: We use the known fact that lines drawn from a vertex of a triangle to its orthocentre and its circumcentre are isogonals which implies that the angles between these lines and the respective sides are equal.
As AD and AO are the altitude and the circumradius of the triangle ABC, both drawn from vertex A, we have $\angle BAO = \angle CAD$. But AE bisects the angle CAD and J is the point of intersecton of angle bisectors of the triangle ABO, hence $\angle BAJ = \frac{1}{2}\angle BAO = \frac{1}{2}\angle CAD = \angle EAD$. This shows that rays AJ and AD are isogonals in the triangle ABE. As AD is also an altitude of the triangle ABE, the line AJ must pass through the circumcentre of the triangle ABE. But the circumcentre of the triangle ABE must also lie on the perpendicular bisector of the side AB. As $OA = OB$, the bisector of the angle AOB coincides with the perpendicular bisector of the side AB. Hence the circumcentre of the triangle ABE is the point J of intersection of lines AJ and OJ. Consequently, $JB = JE$, implying that the triangle JBE is isosceles.
Solution 4: As $OA = OB$, the bisector of the angle AOB coincides with the perpendicular bisector of the side AB. Hence the point J lies on the perpendicular bisector of the side AB, implying that points A and B lie on some circle ω with centre J.
We show that $\angle AJB = 2\angle AEB$. Indeed,

$$\begin{aligned}\angle AEB &= \angle AED = 90^\circ - \angle EAD \\ &= 90^\circ - \tfrac{\angle CAD}{2} = 90^\circ - \tfrac{90^\circ - \angle ACD}{2} = 45^\circ + \tfrac{\angle ACD}{2}.\end{aligned}$$

As $JA = JB$ implies $\angle JAB = \angle JBA$, we obtain

$$\begin{aligned}\angle AJB &= 180^\circ - 2\angle JAB = 180^\circ - \angle OAB \\ &= 180^\circ - \tfrac{180^\circ - \angle AOB}{2} = 90^\circ + \tfrac{\angle AOB}{2} = 90^\circ + \angle ACB = 2\angle AEB.\end{aligned}$$

Hence ω is the circumcircle of the triangle ABE. Consequently, $JB = JE$, implying that the triangle JBE is isosceles.

O13 *(Seniors.)* A gardener Andres wants to plant one currant bush to each cell of his garden of shape 24×2024. He wants to plant as many blackcurrant bushes as possible under the following conditions: There must be at least one redcurrant and at least one whitecurrant bush, and for any cell with a blackcurrant bush, cells that have a common side with it must contain equally many redcurrant and whitecurrant bushes (maybe also 0 of both). Find the largest number of blackcurrant bushes Andres can plant.

Answer: 46632.

Solution: Firstly, we show that 46632 blackcurrant bushes is possible. Let's alternate redcurrant and whitecurrant bushes on the falling diagonal starting from the top left corner; when we reach the bottom edge of the garden, skip one column and proceed similarly along a rising diagonal starting from the next column, and so on. Into all remaining cells, we put blackcurrant bushes (Figure 14 depicts the beginning of the garden).

This way, blocks consisting of 24 columns, each containing 23 blackcurrant bushes and 1 non-blackcurrant bush, alternate with singleton columns that contain blackcurrants only. As $2024 = 80 \cdot 25 + 24$, the last diagonal containing non-blackcurrant bushes ends in the bottom right corner of the garden (no column of blackcurrants follows). Each cell next to a diagonal of non-blackcurrant bushes (including the utmost cell of each column of blackcurrants) has exactly one red and one white neighbour. Other blackcurrant cells have only black neighbours. There are $2024 \cdot 24 - 81 \cdot 24 = 46632$ blackcurrant bushes in total.

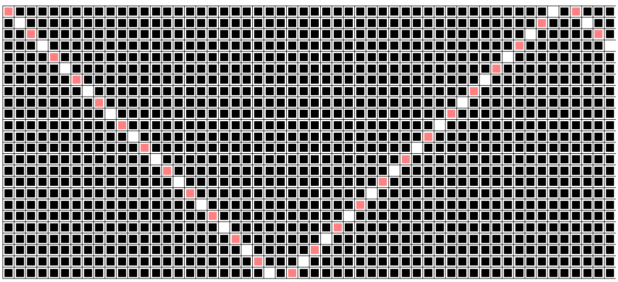


Fig. 14

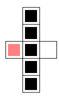


Fig. 15

Secondly, we show that there cannot be more blackcurrant bushes. To this end, consider an arbitrary situation that satisfies the conditions. Let the garden have 24 rows and 2024 columns.

If there is a column containing blackcurrants solely, there must definitely exist two consecutive columns, one of which contains non-blackcurrant bushes and the other does not. Then in the latter column, the blackcurrant bush next to a non-blackcurrant bush of the former column must have another non-black neighbour on the other side (Fig. 15). Thus there cannot be two consecutive columns with blackcurrant bushes only, nor can the first or the last column contain blackcurrants only.

Consider now a block of m consecutive columns, each containing at least one non-black cell. Assume that this block cannot be extended to obtain a bigger block with the same property. We show that if $m < 24$ then this block must contain at least $m + 1$ non-black cells in total. Consider two cases:

- If there is a row that contains no non-black cells within the block then there must be two consecutive rows, one of which contains non-black cells within the block and the other does not. Like above, we see that in the latter row, the black cell next to a non-black cell of the former row must have another non-black neighbour on the other side. Thus the column of this cell must contain two non-black cells. As every column of the block contains at least one non-black cell, the block must contain at least $m + 1$ non-black cells in total.
- If every row contains non-black cells within the block then the block contains at least 24 non-black cells in total. As $m < 24$, this implies that the block contains at least $m + 1$ non-black cells.

Let's add one more column of blackcurrants to the beginning of the garden; by the above, this does not give rise to two consecutive columns of blackcurrants. (We do not apply the condition of the problem to the extra column, it is introduced for convenience only.) Divide the garden into blocks, each of which begins with a column of blackcurrants and ends with the last column before the next column of blackcurrants (or with the last column of the garden if no columns of blackcurrants follow). By the definition of blocks, each block of m columns contains at least $m - 1$ non-black cells; but if $m < 25$ then the block must contain at least m non-black cells. Since $\frac{2025}{25} = 81$, the number of blocks with 1 less non-black cells than columns can be at most 81. Thus the garden contains at least $2025 - 81$ non-black cells and at most $2024 \cdot 24 - (2025 - 81) = 46632$ black cells.

O14 *(Seniors.)* Is there a positive integer n such that 88 divides $2^n + n^3$?

Answer: Yes.

Solution 1: Taking $n = 10$ gives $2^n + n^3 = 1024 + 1000 = 2024 = 88 \cdot 23$, so 88 divides $2^{10} + 10^3$.

Solution 2: We consider divisibility by 11 and by 8 separately. By Fermat's little theorem, we have $2^{10} \equiv 1 \pmod{11}$, whereas $10^3 \equiv (-1)^3 = -1 \pmod{11}$. In summary $2^{10} + 10^3 \equiv 1 - 1 = 0 \pmod{11}$, meaning that 11 divides $2^{10} + 10^3$. As 8 divides both 2^{10} and 10^3, it divides the sum $2^{10} + 10^3$. Thus 88 divides $2^{10} + 10^3$, as desired.

O15 *(Seniors.)* Let $n \geqslant 3$ be an integer. In an $n \times n$ grid there are three invisible monsters: one in the upper right corner square and one in each of

its neighboring squares. In a 2×2 area in the opposite corner of the grid some frogs are placed. A square may contain more than one frog.
The frogs and the monsters take turns making the following moves. On the frogs' turn, each frog makes a knight move, jumping either two squares up and one to the right or one square up and two to the right. On the monsters' turn, they may make up to a total of 3 knight moves, jumping either two squares down and one to the left or one square down and two to the left. The frogs start. If a frog and a monster ever land on the same square, the monster will eat the frog.
Find the least number of frogs needed to ensure that at least one frog could make it to a spot where both possible jumps would take it off the grid.
Answer: 1.
Solution: We color the square in 3 colors by descending diagonals (in Fig. 16 the colors are denoted by A, B, C and X, Y, Z, which are still the same colors, but their order depends on the value of n). Notice that both frogs and monsters can only jump on squares of the same color as the one they started from. As the monsters cover only two colors of squares, but a 2×2 area contains squares of all three colors, it is sufficient to place one frog on a square of the color not covered by the monsters. In this case, no matter the moves, the frog will be safe.

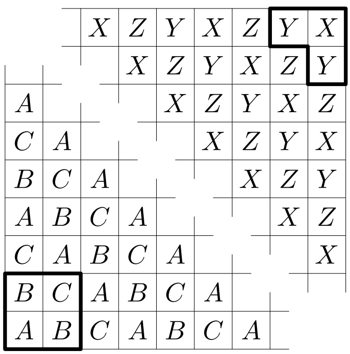


Fig. 16

O16 *(Seniors.)* Find all triples (x, y, z) of positive integers, such that

$$3 \cdot x! + 4 \cdot y! = 5 \cdot z!.$$

Answer: $(2, 1, 2)$ and $(2, 3, 3)$.
Solution 1: If $x \leqslant y$, then $4 \cdot y! < 3 \cdot x! + 4 \cdot y! \leqslant 7 \cdot y!$, so $\frac{4}{5} \cdot y! < z! < \frac{7}{5} \cdot y!$. Two distinct factorials differ by a factor of at least 2, so $z < y$ would yield $z! \leqslant \frac{1}{2} \cdot y! < \frac{4}{5} \cdot y! < z!$, contradiction. Analogously $z > y$ would yield $z! \geqslant 2 \cdot y! > \frac{7}{5} \cdot y! > z!$, contradiction. Therefore the only option is $z = y$. Then the given equation simplifies to $3 \cdot x! = y!$. Here $y > x$ and the product

of $x+1, x+2, \dots, y$ must be 3. This can only happen when $y = x+1 = 3$, giving the solution $(x,y,z) = (2,3,3)$.
If $x > y$, then $3 \cdot x! < 3 \cdot x! + 4 \cdot y! < 7 \cdot x!$, so $\frac{3}{5} \cdot x! < z! < \frac{7}{5} \cdot x!$. Analogously to the previous case we get $z = x$. Then the given equation simplifies to $4 \cdot y! = 2 \cdot x!$ or $2 \cdot y! = x!$. Analogously to the previous case, this is only possible when $x = y+1 = 2$, giving the solution $(x,y,z) = (2,1,2)$.
Solution 2: Notice that $x \leqslant z$ and $y \leqslant z$, because if either $x \geqslant z+1$ or $y \geqslant z+1$, then

$$5 \cdot z! = 3 \cdot x! + 4 \cdot y! \geqslant 3 \cdot \max(x!, y!) \geqslant 3 \cdot (z+1)!,$$

where dividing by $z!$ gives $5 \geqslant 3(z+1)$, from which $z \leqslant \frac{2}{3} < 1$, contradiction.
We consider three cases.

- If $x = z$, then like in Solution 1, we only get the solution $(2,1,2)$.
- If $y = z$, then like in Solution 1, we only get the solution $(2,3,3)$.
- If $x \leqslant z-1$ and $y \leqslant z-1$, then

  $$5 \cdot z! = 3 \cdot x! + 4 \cdot y! \leqslant 3 \cdot (z-1)! + 4 \cdot (z-1)! = 7 \cdot (z-1)!,$$

  where dividing by $(z-1)!$ gives $5z \leqslant 7$. This leaves only the option $z = 1$, which is impossivle by the assumptions $x \leqslant z-1$ and $y \leqslant z-1$.

Thus the only suitable triples are $(2,1,2)$ and $(2,3,3)$.
Solution 3: We divide the sides $3x! + 4y! = 5z!$ by the smallest factorial present in the equation. This leaves an equation $3a + 4b = 5c$, where a, b, c are positive integers, of which at least one is equal to 1.
If $c = 1$, then $3a + 4b = 5$, giving no solutions. If $a = b = 1$, then $7 = 5c$, also giving no solutions.
If $a = 1$, $b > 1$, $c > 1$ and $3 + 4b = 5c$, then $y > x$ and $z > x$, meaning that both b and c are divisible by $x+1$. Then 3 is also divisible by $x+1$, which gives $x = 2$. Then $z = 3$, or else the right hand side of the equation is divisible by 4, whereas the left hand side isn't. Then also $y = 3$.
If $b = 1$, $a > 1$, $c > 1$ and $3a + 4 = 5c$, then $x > y$ and $z > y$, meaning that both a and c are divisible by $y+1$. Then 4 is also divisible by $y+1$, which gives $y = 1$ or $y = 3$. If $y = 1$, then $z = 2$, or else the right hand side of the equation is divisible by 3, whereas the left hand side isn't. Then also $x = 2$. But if $y = 3$, then the left hand side is never divisible by 5, so there are no solutions.
Thus the only suitable triples are $(2,1,2)$ and $(2,3,3)$.

O17 *(Seniors.)* Mama snail and her child want to visit a neighbour who lives at distance 75 cm. Every hour, they have planned to use 45 minutes to move and 15 minutes to rest. On the n-th hour, they move $\frac{1}{n^2+1}$ metres forward, but instead of resting, the child pulls them backwards by $\frac{1}{n+1}$ of this hour's distance. Will they ever reach the neighbour, and if so, when?

Answer: No.
Solution: Combining both parts of the n-th hour, the total distance travelled forward is $\frac{1}{n^2+1} - \frac{1}{n+1} \cdot \frac{1}{n^2+1} = \frac{n}{(n+1)(n^2+1)}$ metres. On the first hour, this means $\frac{1}{4}$ metres. Notice that $\frac{n}{(n+1)(n^2+1)} < \frac{n}{(n+1)n^2} = \frac{1}{(n+1)n} = \frac{1}{n} - \frac{1}{n+1}$. Thus by the end of the m-th hour, where $m \geqslant 2$, they have travelled less than $\frac{1}{4} + \left(\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{m} - \frac{1}{m+1}\right)\right) = \frac{3}{4} - \frac{1}{m+1}$ metres, which is less than 75 centimetres. Thus they will never reach the neighbour by the end of the hour.
We will show that they will also not reach the neighbour between the two parts of an hour. Indeed, 45 minutes after starting they have gone $\frac{1}{2}$ metres and 1 hour and 45 minutes after starting $\frac{1}{4} + \frac{1}{5}$ metres, both less than 75 centimetres. But if $m \geqslant 2$, then m hours and 45 minutes after starting they have travelled less than $\frac{3}{4} - \frac{1}{m+1} + \frac{1}{(m+1)^2+1}$ metres, which is still less than 75 centimetres, as $\frac{1}{(m+1)^2+1} < \frac{1}{m+1}$.
Thus they will never reach the neighbour.

O18 *(Seniors.)* In an acute triangle ABC, the extension of the altitude AD over D intersects the circumcircle at E. The midpoint of CE is F. The circumcircles of ABC and DEF intersect at $G \neq E$. The foot of the altitude drawn from A to FG is P. Prove that $DA = DP$.
Solution 1: We first show that $\angle BGF = 90^\circ$ (Fig. 17). For this we notice that F as the midpoint of the hypothenuse in CDE is also its circumcenter, so $FC = FD$. Together with $BEGC$ being cyclic, we get

$$\angle EGB = \angle ECB = \angle FCD = \angle FDC.$$

Thus

$$\begin{aligned}\angle BGF &= \angle EGF - \angle EGB \\ &= 180^\circ - \angle EDF - \angle FDC = 180^\circ - \angle EDC = 180^\circ - 90^\circ = 90^\circ.\end{aligned}$$

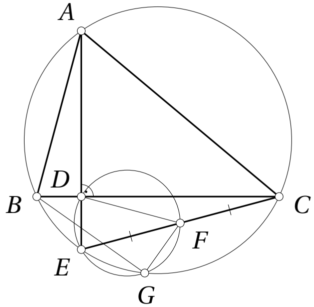


Fig. 17

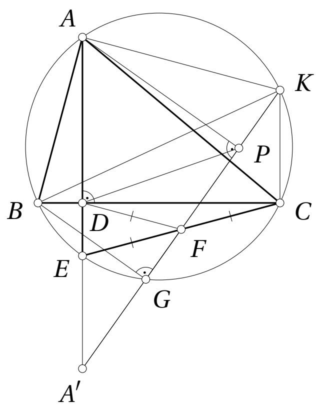


Fig. 18

Let A' and K be the intersections of GF with AD and with the circumcircle of ABC, respectively (Fig. 18). As $\angle BGK = \angle BGF = 90^\circ$, we have that BK is a diameter of the circumcircle of ABC. Then also $\angle BCK = 90^\circ$, from which $CK \parallel AE$. Now $AKCE$ as a cyclic quadrilateral with two parallel sides must be an isosceles trapezium. Combining this with $FD = FE$, we get $\angle EDF = \angle DEF = \angle AEC = \angle EAK$, so $DF \parallel AK$. From $FE = FC$ and the parallel lines we get equal triangles FCK and FEA', from which $FA' = FK$. Thus DF is a midline of AKA', from which D is the midpoint of AA'. In conclusion D is the circumcenter of the right-angled triangle APA', which finishes the proof.

Solution 2: Let A' be the reflection of A over BC and let BK be a diameter of the circumcircle of ABC.

First, we will show that F lies on $A'K$ (Fig. 19). As $KC \perp BC$ and $AD \perp BC$, we have $KC \parallel EA'$. Also

$$\begin{aligned} \angle AEK &= \angle ACK = 90^\circ - \angle ACB, \\ \angle AA'C &= \angle DA'C = \angle DAC = \angle 90^\circ - \angle ACB. \end{aligned}$$

Thus $\angle AEK = \angle AA'C$, from which $EK \parallel A'C$. Therefore $A'CKE$ is a parallellogram. As the intersection of the diagonals of a parallellogram divides both of its diagonals in half and F is the midpoint of CE, it must also be the midpoint of $A'K$, so A', K, F are indeed collinear.

Now we show that G also lies on $A'K$. For this, let G' be the second intersection of $A'K$ and the circumcircle of ABC (Fig. 20). Notice that DF is a midline of $A'AK$, so $DF \parallel AK$. Therefore

$$\angle EDF = \angle EAK = 180^\circ - \angle EG'K = 180^\circ - \angle EG'F,$$

which shows that G' lies on the circumcircle of DEF, meaning that $G' = G$. We have shown that A' lies on GF, which also contains P. Thus APA' is a right triangle, whose circumcenter is the midpoint D of AA'. Therefore $DA = DP$, as desired.

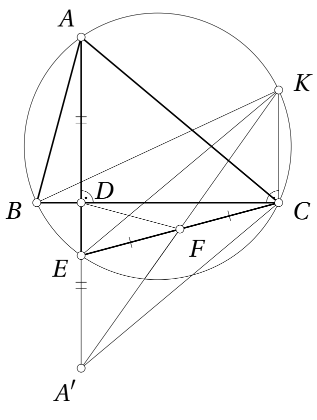


Fig. 19

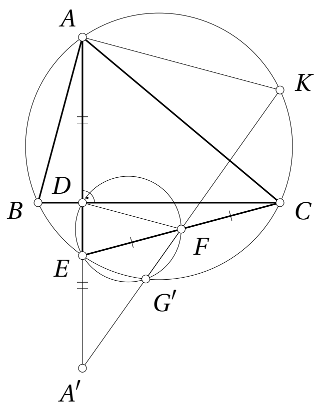


Fig. 20

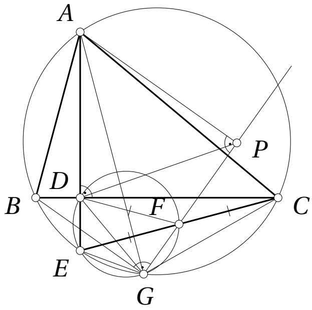


Fig. 21

Solution 3: Like in Solution 1, we show that $FC = FD$ and $BG \perp GF$. Since also $AP \perp GF$ (Fig. 21), we have $AP \parallel BG$. From this

$$\angle GAP = \angle AGB = \angle ACB = \angle ACD,$$

so the right triangles GAP and ACD are similar. Therefore

$$\frac{GA}{AC} = \frac{PG}{DA}. \tag{7}$$

On the other hand, we have the equalities

$$\angle GDF = \angle GEF = \angle GEC = \angle GAC,$$
$$\angle FGD = \angle FED = \angle CEA = \angle CGA,$$

from which it follows that the triangles GDF and GAC are similar. Thus

$$\frac{GA}{AC} = \frac{GD}{DF}. \tag{8}$$

From (7) and (8) we get $\frac{PG}{DA} = \frac{GD}{DF}$ or

$$\frac{GD}{PG} = \frac{DF}{DA}. \tag{9}$$

Notice also that there must exist a spiral similarity at G between the similar triangles GDF and GAC. Hence AGD and CGF are also similar, implying

$$\frac{GC}{GA} = \frac{FC}{DA}. \tag{10}$$

As $DF = FC$, the right hand sides of (9) and (10) are equal. So the left hand sides must be equal as well, i.e., $\frac{GD}{PG} = \frac{GC}{GA}$. As $\angle PGD = \angle FGD = \angle CGA$, triangles PGD and AGC must also be similar. Therefore

$$\frac{GA}{AC} = \frac{PG}{DP}. \tag{11}$$

Combining (7) and (11) gives $DA = DP$, as desired.

Selected Problems from the Final Round of National Olympiad

F1 *(Grade 7.)* The product abc of positive integers a, b, and c is divisible by 3, and the equations $a = \frac{b^2}{2} = \frac{c}{4}$ hold. Find the smallest possible sum of the numbers a, b, and c under these conditions.
Answer: 96.
Solution: The given equations are equivalent to $2a = b^2 = \frac{c}{2}$. So b^2 is divisible by 2, therefore b is also divisible by 2. Since the product abc is divisible by 3, one of the numbers a, b and c must be divisible by 3. If a is divisible by 3, then b^2 is also divisible by 3, therefore b must be divisible by 3. If c is divisible by 3, then b^2 must also be divisible by 3. So in all cases b is divisible by 3. Therefore b is divisible by both 2 and 3, so b must be at least $2 \cdot 3 = 6$. If $b = 6$, then $a = \frac{b^2}{2} = 18$ and $c = 2b^2 = 72$ and $a + b + c = 96$. If b is larger, then b^2 and also a and c would be larger, hence the sum found is the smallest possible.

F2 *(Grade 7.)* The letters M, A, T, E, I and K are assigned the numbers 1, 2, 3, 4, 5 and 6 in a certain order so that different letters correspond to different numbers. The sum of the numbers corresponding to the letters of the word MATEMAATIK (taking into account repetitions) is 42 and the sum of the numbers corresponding to the letters of the word KEEMIK (taking into account repetitions) is 13. Find the sum of the numbers corresponding to the letters of the word IT.
Answer: 8.
Solution: In the words MATEMAATIK, KEEMIK and IT, all the given letters occur exactly 3 times. Since the sum of the numbers corresponding to the letters M, A, T, E, I and K is $1 + 2 + 3 + 4 + 5 + 6 = 21$, the sum of the numbers corresponding to the letters in these three words is $3 \cdot 21 = 63$. Consequently, the sum of the numbers corresponding to the letters in the word IT is $63 - 42 - 13 = 8$.

F3 *(Grade 7.)* The diagonals of the quadrilateral $ABDE$ meet at C. The segments AB and CE are of equal length 8 cm, and the segments AE and CD are also of equal length. The perimeter of the triangle CDE is 35 cm. Given that $\angle BAC = \angle AEC$, find the perimeter of the pentagon $ABCDE$.
Answer: 54 cm.
Solution: Using the condition of the problem, we get

$$\angle BAE = \angle BAC + \angle CAE = \angle AEC + \angle CAE$$

(Fig. 22). From the triangle ACE we get $\angle AEC + \angle CAE = \angle DCE$. Thus $\angle BAE = \angle DCE$. At the same time, $AE = CD$ and $AB = CE$. Consequently, the triangles AEB and CDE are equal, hence the perimeter of the triangle AEB is 35 cm.

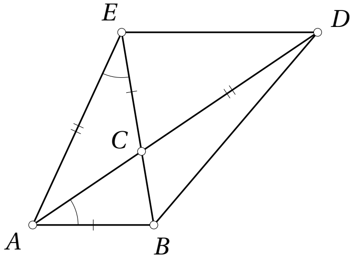


Fig. 22

Now we get

$$\begin{aligned} EA+AB+BC+CD+DE &= (EA+AB+BE-CE)+(CD+DE+EC-CE) \\ &= (EA+AB+BE)+(CD+DE+EC)-2CE. \end{aligned}$$

Since $EA + AB + BE = CD + DE + EC = 35$ cm and $CE = 8$ cm, the perimeter of the pentagon $ABCDE$ is $2 \cdot 35$ cm $- 2 \cdot 8$ cm $= 54$ cm.

Remark: The shape and dimensions of the pentagon $ABCDE$ are uniquely determined by the conditions of the problem.

F4 *(Grade 7.)* In how many ways can one choose 5 numbers from the list

$$\frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4, 5, 6, 7, 8$$

so that the product of the chosen numbers is 1?

Remark: Two choices are considered different if one choice contains a number that the other choice does not. The order of numbers is not important.

Answer: 41.

Solution 1: Let's first count the choices that include the number 1. If some three of the remaining numbers were 2, 3, and 4, then the conditions of the problem would be violated, because the denominators of all the fractions in the list are smaller than $2 \cdot 3 \cdot 4$, or 24, and the product of the selected 5 numbers could not be 1. If instead of 2, 3, and 4 some other 3 integers are chosen, their product will be even larger, so the conditions of the problem cannot be met. The situation is similar if 3 fractions are chosen next to the number 1. Thus, in addition to the number 1, the choice must include 2 integers and 2 fractions, with the product of the integers equal to the product of the denominators of the fractions. There are exactly 21 ways to choose 2 integers in addition to the number 1, and the resulting products are in the following table:

	3	4	5	6	7	8
2	6	8	10	12	14	16
3		12	15	18	21	24
4			20	24	28	32
5				30	35	40
6					42	48
7						56

We see that the numbers 12 and 24 appear in the table 2 times, the remaining numbers are unique. Thus, we get 21 options where the selected integers are the same as the denominators of the selected fractions, and in addition 4 options where the selected integers and the denominators of the selected fractions are not the same, but their product is either 12 or 24. In total, there are 25 options with the number 1.

Now we count the options that do not have the number 1. We have to choose either 3 integers and 2 fractions or 3 fractions and 2 integers. Since, due to symmetry, there are the same number of options of both types, we count the options involving 3 integers and 2 fractions. If 2 were not included in the selection, the product of the selected integers should be at least $3 \cdot 4 \cdot 5 = 60$, but according to the multiplication table, the product of the denominators of any 2 fractions is less than 60, which is why the condition of the problem cannot be met. Therefore, 2 must be included in the selection. Similarly, we see that either 3 or 4 must also be included in the selection. We get the possibilities

$$2 \cdot 3 \cdot 4 = 24, \quad 2 \cdot 3 \cdot 5 = 30, \quad 2 \cdot 3 \cdot 6 = 36, \quad 2 \cdot 3 \cdot 7 = 42, \quad 2 \cdot 3 \cdot 8 = 48,$$
$$2 \cdot 4 \cdot 5 = 40, \quad 2 \cdot 4 \cdot 6 = 48, \quad 2 \cdot 4 \cdot 7 = 56, \quad 2 \cdot 4 \cdot 8 = 64.$$

Of these 9 products, only 36 and 64 do not appear in the multiplication table above, the product 24 appears there 2 times and the remaining 6 products 1 time. So we get $2 \cdot 0 + 1 \cdot 2 + 6 \cdot 1 = 8$ possibilities. There are the same number of possibilities with 3 fractions and 2 integers. So there are $2 \cdot 8 = 16$ possibilities without the number 1.

Hence the total number of possibilities is $25 + 16 = 41$.

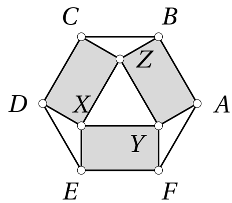
Solution 2: The list of numbers contains 7 pairs of reciprocals in the form $\left(\frac{1}{a}, a\right)$. At most 2 such pairs can occur among the 5 numbers chosen.

If there are two pairs of reciprocals in the choice, then the last number must be 1. To choose 2 pairs from the 7 existing pairs of reciprocals, there are $\frac{7 \cdot 6}{2} = 21$ possibilities.

If there is one pair of reciprocals in the choice, then the product of the remaining 3 numbers must be 1. If the integers a and b are among these 3 numbers, then the last number must be $\frac{1}{ab}$, so the integer ab is also in the original list. Similarly if the fractions $\frac{1}{a}$ and $\frac{1}{b}$ are among these 3 numbers, then a, b and ab must be in the original list. The only such numbers are $2, 3, 6$ and $2, 4, 8$, each of which gives 2 choices for the 3 numbers from the original list. Then there are $7 - 3 = 4$ options for the pair of reciprocal numbers. So in this case there are $2 \cdot 2 \cdot 4 = 16$ possibilities.

If there are no pairs of reciprocals in the selection, then each integer can appear in the selection at most once, either by itself or in the denominator of the reciprocal. The prime numbers 5 and 7 cannot appear in either role. The prime number 3 only appears as a factor in the numbers 3 and 6; if both of them were omitted, there would not be enough numbers left in the selection, so they must both be in the selection, either as 3 and $\frac{1}{6}$ or as $\frac{1}{3}$ and 6. The product of these numbers is $\frac{1}{2}$ or 2, respectively. The remaining numbers are 1, 2, 4, and 8, from which 3 must be chosen. If one does not choose 1, then it is not possible to get the number 2 or $\frac{1}{2}$ by using each remaining number or its reciprocal as a factor only once. Thus, the number 1 must be in the selection. From the remaining numbers and their reciprocals, the number 2 or $\frac{1}{2}$ can be composed in 4 ways: $\frac{1}{2} \cdot 4$, $\frac{1}{4} \cdot 8$, $2 \cdot \frac{1}{4}$, and $4 \cdot \frac{1}{8}$. So the total number of choices is $21 + 16 + 4 = 41$.

F5 *(Grade 7.)* Inside a regular hexagon $ABCDEF$, equal rectangles $ABZY$, $CDXZ$, and $EFYX$ are drawn. How much of the area of the hexagon $ABCDEF$ do these rectangles cover?



Answer: $\frac{2}{3}$.

Solution: Each interior angle of a regular hexagon has a size of 120°. Thus, a regular hexagon can be divided into 6 equilateral triangles with side lengths equal to the hexagon itself (Fig. 23). Denoting the area of such a triangle as S, the area of the hexagon $ABCDEF$ is therefore $6S$.

The opposite sides of a rectangle are of equal length. Hence $XY = EF$, $YZ = AB$ and $XZ = CD$, so XYZ is also an equilateral triangle with the same side length. Since the rectangles $ABZY$, $CDXZ$ and $EFYX$ are equal, the triangles AYF, BZC and DXE are isosceles. Their bases are the sides of the hexagon, the base angle is $120° - 90° = 30°$ and the vertex angle is $180° - 2 \cdot 30° = 120°$. Thus, it is possible to form one equilateral triangle from these three triangles, whose side length is equal to the side length of the hexagon (Fig. 24). In total, the area not covered by the rectangles is $2S$. Therefore, the total area of the rectangles is $6S - 2S = 4S$, which is $\frac{2}{3}$ of the area of the hexagon.

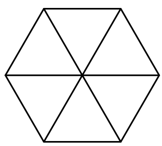


Fig. 23

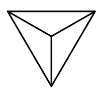


Fig. 24

F6 *(Grade 8.)* Find all natural numbers whose last digit is not zero and deleting the first digit of which gives a number exactly 25 times smaller.

Answer: 625, 3125, 9375.

Solution 1: Let n be the desired number, and let m be the number obtained by deleting the first digit. Since the last digit of the product $m \cdot 25$ is not 0, it can only be 5. Since $5 \cdot 25$ has three digits, it cannot be the number n itself,

so the number m must have more digits. Consider a digit d in both n and m. Then in the product d is obtained from multiplying all lower-order digits by 25 and adding the last digit of the number $d \cdot 5$ to the corresponding digit in the product. If d is even then the corresponding digit in the product of lower order digits by 25 is d, i.e. an even number, while in the case of odd d, it is $d - 5$ or $d + 5$, i.e. again an even number.
Since the tens digit of $5 \cdot 25$ is 2, the tens digit of n and m must be either 2 or 7. If it is 2 and m does not have more digits, then $n = 25 \cdot 25 = 625$ which satisfies the conditions of the problem. If there are more digits in the number m, then the hundreds digit of the numbers m and n must be 6 or 1, because the hundreds digit of the intermediate result of multiplying lower-order numbers is 6. Now n cannot be $625 \cdot 25$ because the product has 5 digits. Since the thousands digit 5 of this product is odd, it is not possible to add more digits to the number m. If the hundreds digit is 1 then we get the possibility $m = 125$ and $n = 3125$, which satisfies the conditions of the problem, and since the thousands digit in this number is odd, it is not possible to add more digits to the number m here either.
If the tens digit of m and n is 7, then n cannot be $75 \cdot 25$ because the product has 4 digits. Since the hundreds digit of $75 \cdot 25$ is 8, the hundreds digit of m can be 8 or 3. Again n cannot be $875 \cdot 25$ because the product has 5 digits, and since the thousands digit 1 of the product is odd, it is not possible to add more digits to the number m. If the hundreds digit of m is 3 we get $m = 375$ and $n = 9375$, which satisfies the conditons of the problem. Since the thousands digit of the product is odd, it is not possible to add more digits to the number m here either.
Thus only 625, 3125 and 9375 satisfy the conditions of the problem.
Solution 2: Let n be the desired number. Let the first digit of this number be a and the number formed from the remaining digits be m, with the number m having k digits. Then $n = 10^k a + m = 25m$, implying $10^k a = 24m$. Thus, the number $10^k a = 2^k 5^k a$ is divisible by the number 24. Consequently, the number $2^k 5^k a$ must be divisible by the numbers 2^3 and 3. Since the number $2^k 5^k$ is not divisible by 3, the number a must be divisible by 3, which leaves the possibilities $a = 3$, $a = 6$ and $a = 9$.

- If $a = 3$, then $2^k 5^k$ must be divisible by 2^3. Therefore $k \geqslant 3$ and we get $m = \frac{10^k a}{24} = \frac{3000 \cdot 10^{k-3}}{24} = 125 \cdot 10^{k-3}$. Since last digit of m is not zero, we have $m = 125$, whence $n = 3125$.
- If $a = 6$, then $2^k 5^k$ must be divisible by 2^2. Therefore $k \geqslant 2$ and we get $m = \frac{10^k a}{24} = \frac{600 \cdot 10^{k-2}}{24} = 25 \cdot 10^{k-2}$. Since the last digit is not zero, we have $m = 25$, whence $n = 625$.
- if $a = 9$, then $2^k 5^k$ must be divisible by 2^3. Therefore $k \geqslant 3$ and we get $m = \frac{10^k a}{24} = \frac{9000 \cdot 10^{k-3}}{24} = 375 \cdot 10^{k-3}$. Since the last digit of m is not zero, we have $m = 375$, whence $n = 9375$.

F7 *(Grade 8.)* Let ABC be a triangle and P a point inside it. Let A', B', C' be the reflections of A, B, C from point P, respectively. Find the ratio of the areas of the hexagon $AB'CA'BC'$ and the triangle ABC.
Answer: 2.
Solution: Denote the area of the figure $\mathcal{K}$ by $S_{\mathcal{K}}$. Note (Fig. 25) that

$$S_{AB'CA'BC'} = S_{APB'} + S_{B'PC} + S_{CPA'} + S_{A'PB} + S_{BPC'} + S_{C'PA}.$$

Furthermore, $S_{APB'} = S_{APB}$, because triangles APB' and APB have the same altitude drawn from vertex A and the corresponding bases are equal. Similarly we get $S_{B'PC} = S_{BPC}$, $S_{CPA'} = S_{CPA}$, $S_{A'PB} = S_{APB}$, $S_{BPC'} = S_{BPC}$ and $S_{C'PA} = S_{CPA}$. Hence

$$S_{AB'CA'BC'} = 2S_{APB} + 2S_{BPC} + 2S_{CPA} = 2\left(S_{APB} + S_{BPC} + S_{CPA}\right) = 2S_{ABC}.$$

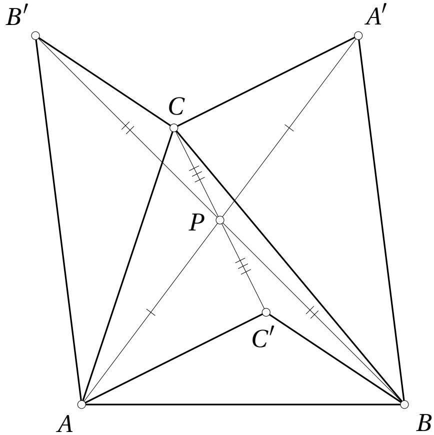


Fig. 25

F8 *(Grade 8.)* The price of the old model of smartwatch A differs from the price of the old model of smartwatch B by $p\%$ ($0 < p < 100$). The new model of watch A is $q\%$ more expensive than the old model of watch A, and the new model of watch B is $q\%$ cheaper than the old model of watch B ($0 < q < 100$). The price of the new model of watch A differs from the price of the new model of watch B by $p\%$.
(a) Is the new model of watch B more expensive, cheaper, or just as expensive as the old model of watch A?
(b) Is the new model of watch A more expensive, cheaper, or just as expensive as the old model of watch B?
Answer: (a) Just as expensive; (b) Cheaper.
Solution: Since the new model of watch A is more expensive than the old model of watch A and the new model of watch B is cheaper than the old model of watch B, but the difference in price is still $p\%$, the old model

of watch A must be $p\%$ cheaper than the old model of watch B, and the new model of watch A must be $p\%$ more expensive than the new model of watch B.

Let the price of the old model of watch B be x. Then the price of the new model of watch B is $\left(1-\frac{q}{100}\right)x$. The old model of watch A then costs $\left(1-\frac{p}{100}\right)x$, while the new model of watch A costs $\left(1+\frac{q}{100}\right)\left(1-\frac{p}{100}\right)x$. Since from the previous paragraph the price of the new model of watch A is $\left(1+\frac{p}{100}\right)\left(1-\frac{q}{100}\right)x$, we have

$$\left(1+\frac{q}{100}\right)\left(1-\frac{p}{100}\right)x=\left(1+\frac{p}{100}\right)\left(1-\frac{q}{100}\right)x.$$

After cancelling x from both sides, opening the parentheses and simplifying, we get $p=q$.

(a) Since $p=q$, the prices of the new model of watch B and of the old model of watch A are both $\left(1-\frac{p}{100}\right)x$.

(b) Since $p=q$, the price of the new model of watch A is $\left(1-\frac{p^2}{10000}\right)x$ which is less than the price of the old model of watch B.

F9 *(Grade 8.)* The midpoint of the hypotenuse AB of a right-angled triangle ABC is M. A point D lies on the side BC such that the circumcircle of the triangle ACD intersects the line DM at some point K between the points D and M. Let L be the reflection of the point K from the point M. The circumcircles of the triangles ACD and LBC intersect at the point N, $N\neq C$. Find the size of the angle KNL.

Answer: 90°.

Solution 1: The right angle DCA subtends the chord AD of the circumcircle of triangle ACD (Fig. 26), therefore AD is a diameter of this circle. Since point K lies on the same circle, $\angle AKD=90^\circ$. From supplementary angles we get $\angle MKA=180^\circ-\angle AKD=90^\circ$.

From the conditions of the problem $AM=MB$ and $KM=ML$ and from the vertex angles we get $\angle AMK=\angle BML$. Thus, the triangles AMK and BML are equal. Consequently $\angle MLB=\angle MKA=90^\circ$.

Next, we show that $\angle NLK=\angle NKA$. It suffices to show the equality $\angle NLB=\angle NKD$ since $\angle NLK=\angle NLB-\angle KLB=\angle NLB-90^\circ$ and

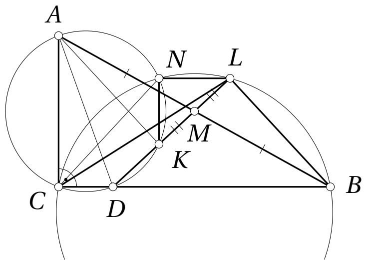


Fig. 26

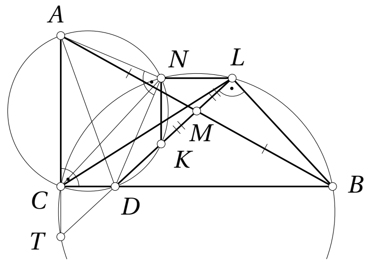


Fig. 27

$\angle NKA = \angle NKD - \angle AKD = \angle NKD - 90°$. Furthermore, since N, L, B and C lie on the same circle in this order, we have $\angle NLB = 180° - \angle BCN$, while the points N, K, D and C lying on one circle in this order implies $\angle NKD = 180° - \angle DCN = 180° - \angle BCN$. So $\angle NLK = \angle NKA$ as desired. Hence $\angle NLK + \angle LKN = \angle NLK + \angle LKA - \angle NKA = \angle LKA = 90°$, which implies $\angle KNL = 180° - 90° = 90°$.

Solution 2: As in Solution 1 we show that AD is a diameter of the circumcircle of triangle ACD and $\angle MLB = 90°$. Since point N lies on the circumcircle of triangle ACD we have $\angle AND = 90°$.

Let T be the intersection of the lines LD and AC (Fig. 27). From supplementary angles we get $\angle TCB = 180° - \angle BCA = 90°$. Since $\angle TLB = 90°$, points C and L lie on the circle with the diameter TB, whence T lies on the circumcircle of triangle LBC.

We show that $\angle DNL = \angle ANK$. Since the points C, N, L, T lie on the same circle in this order, we have $\angle CNL + \angle LTC = 180°$. Hence

$$\begin{aligned}\angle DNL &= \angle CNL - \angle CND = 180° - \angle LTC - \angle CND \\ &= 180° - \angle DTA - \angle TAD.\end{aligned}$$

From triangle TAD we get $180° - \angle DTA - \angle TAD = \angle ADT$ and from supplementary angles $\angle ADT = 180° - \angle KDA$. Since A, N, K, D lie on the same circle in this order, we get $\angle 180° - \angle KDA = \angle ANK$. So indeed $\angle DNL = \angle ANK$.

Finally notice that $\angle ANK = \angle AND + \angle DNK = 90° + \angle DNK$, hence $\angle DNL = 90° + \angle DNK$. Consquently $\angle KNL = \angle DNL - \angle DNK = 90°$.

F10 *(Grade 8.)* A mother has 7 apples, 6 pears, and 5 oranges. She wants to divide them among 2 children so that each gets the same number of fruits. In how many different ways can this be done?

Remark: We consider the distributions of fruit to be different if a child receives a different number of some types of fruit.

Answer: 36.

Solution 1: According to the conditions, each child must receive 9 fruits. It suffices to find how many possibilities there are to give the first child 9 fruits, because the second child receives all the remaining fruits.

The first child can be given 0 to 6 pears and 0 to 5 oranges. Disregarding the condition that he must receive 9 fruit in total, there are a total of $7 \cdot 6$, or 42, possibilities for giving pears and oranges. The possibilities where the first child receives more than 9 of pears and oranges alone, and also those where the first child gets less than 2 of pears and oranges are not suitable. The possibilities where the total number of pears and oranges is more than 9 are 3 ($6 + 4$, $5 + 5$ and $6 + 5$), while the possibilities where the total number of pears and oranges is less than 2 are also 3 ($1 + 0$, $0 + 1$ and $0 + 0$). Thus, there are $42 - 3 - 3 = 36$ suitable possibilities.

Solution 2: Again it suffices to find how many possibilities there are to give the first child 9 fruits.

If the first child gets more apples than the second child, then the first child gets 4 apples and 5 more fruits. Mark with 5 circles the fruits – apples, followed by pears, and finally oranges – and with 2 dashes the places where one kind of fruit changes to another. Then all possible choices of 5 fruits are represented as a word consisting of 7 characters, and the choice consists in determining the positions where the dashes are located. We get $\frac{7 \cdot 6}{2} = 21$ possibilities. But the possibilities where before the first dash there are more than 3 circles are not suitable, because we have only 3 apples left. In this case the dashes are either on fifth and sixth, fifth and seventh or sixth and seventh position. Hence $21 - 3 = 18$ possibilities remain.
The possibilities where the first child gets fewer apples than the second child are symmetrical, so there are the same number of them. So there are a total of $18 \cdot 2 = 36$ different ways to distribute the fruits.

F11 *(Grade 9.)* Call a natural number *twistable* if it does not contain digits 3, 4, 7 and its last digit is not zero. The *twisting* of a twistable number is the number obtained after the following two steps:

- Reverse the order of digits of the given number;
- Twist each digit: 0, 1 and 8 remain unchanged, 2 and 5 are turned into each other, 6 and 9 are turned into each other.

For instance, the twisting of the number 68012 is 51089 and the twisting of the number 69 is 69.
Find all integers that can be represented as the ratio of a twistable positive integer n and its twisting k.
Answer: 1.
Solution: Since the numbers n and k are positive integers of the same length, the quotient $\frac{n}{k}$ must be a single-digit positive number, because multiplying by a multi-digit number increases the number of digits. The quotient 1 is obviously possible (e.g. $\frac{69}{69} = 1$). We show that no other quotient is possible.

- If the first digit of the number k is 1, then the last digit of the number n is 1. The quotient $\frac{n}{k}$ cannot be 2, 4, 5, 6, or 8, because the multiples of these numbers cannot end with the digit 1. If the quotient $\frac{n}{k}$ were 3 or 7, then the last digit of k should be 7 or 3, respectively. However, these digits cannot occur in a twisting. If the quotient $\frac{n}{k}$ were 9, then the last digit of the number k should be 9 and the first digit of the number n should therefore be 6. However, since $n = 9k$, the number n can only start with the digit 9. Therefore, the only possibility is $\frac{n}{k} = 1$.
- If the first digit of the number k is 2, then the last digit of the number n is 5. The quotient $\frac{n}{k}$ cannot be 5, 6, 7, 8, or 9, because in these cases the number n would have more digits than the number k. The quotient $\frac{n}{k}$ cannot be 2 or 4, because the multiples of these numbers cannot end with the digit 5. If the quotient $\frac{n}{k}$ were 3, then the last digit of the number k should also be 5 and the first digit of the number n should

therefore be 2. However, since $n = 3k$, the number n can only start with the digits 6, 7, and 8. So again, the only possibility is $\frac{n}{k} = 1$.

- The first digit of the number k cannot be 3 or 4, because these numbers do not occur in a twisting.
- If the first digit of the number k is 5, 6, 7, 8, or 9, then $\frac{n}{k} \geqslant 2$ is not possible, because then there would be more digits in the number n than in the number k. Thus, the only possibility is $\frac{n}{k} = 1$.

F12 *(Grade 9.)* Prove that the value of the expression

$$x^4 - 2x^3 - 88x^2 + 90x + 2025$$

is positive regardless of the value of x on the real line.
Solution: Using equalities $2025 = 45^2$ and $90 = 2 \cdot 45$ we get

$$\begin{aligned} x^4 - 2x^3 - 88x^2 + 90x + 2025 &= x^4 - 2x^3 - 89x^2 + (x+45)^2 \\ &= \left(x^2\right)^2 - 2x^2(x+45) + (x+45)^2 + x^2 \\ &= \left(x^2 - x - 45\right)^2 + x^2. \end{aligned}$$

But $\left(x^2 - x - 45\right)^2 + x^2 \geqslant 0$. Equality is impossible because it could occur only if $x = 0$ and also $x^2 - x - 45 = 0$. Thus indeed $\left(x^2 - x - 45\right)^2 + x^2 > 0$.

F13 *(Grade 9.)* Let D be a point on the side BC of an acute triangle ABC and let E be a point on the line segment AD. Let F and G be the feet of the altitudes drawn from the vertex D in triangles ABD and ACD, respectively. The line BE intersects the circumcircle of the triangle DEG at point $H \neq E$. Prove that points B, F, G and H are concyclic.
Solution: Since $\angle AFD = \angle AGD = 90^\circ$, the quadrilateral $AFDG$ is cyclic (Fig. 28). Using inscribed angles in the circumcircle of the quadrilateral $AFDG$, we get

$$\angle BFG = 180^\circ - \angle AFG = 180^\circ - \angle ADG = 180^\circ - \angle EDG.$$

According to the conditions of the problem, the points D, E, G, and H are concyclic. Next, we consider three cases.

- If the order of the points is D, E, G, H (Fig. 29), then from the angles inscribed into the circumcircle of the quadrilateral $DEGH$ we obtain

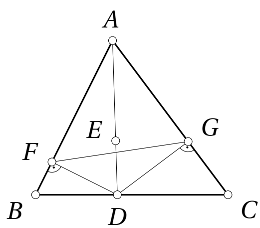


Fig. 28

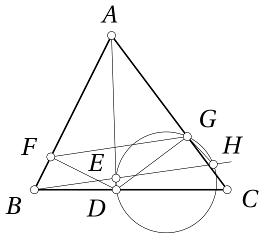


Fig. 29

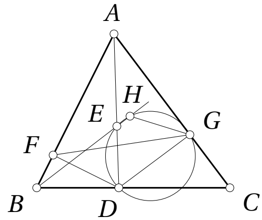


Fig. 30

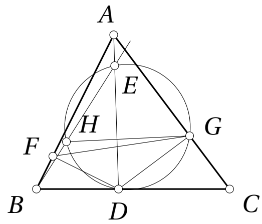


Fig. 31

$\angle BHG = \angle EHG = \angle EDG$. Thus $\angle BFG = 180^\circ - \angle BHG$, so the points B, F, G and H are concyclic.

- If the order of the points is D, E, H, G (Fig. 30), then opposite angles of the cyclic quadrilateral $DEHG$ give $\angle BHG = \angle EHG = 180^\circ - \angle EDG$. So $\angle BFG = \angle BHG$, hence B, F, H and G are concyclic.
- If the order of the points is D, H, E, G (Fig. 31), then we similarly get $\angle BHG = 180^\circ - \angle EHG = 180^\circ - \angle EDG$. So again $\angle BFG = \angle BHG$, hence B, F, H and G are concyclic.

F14 *(Grade 9.)* Ats and Pets take turns to write representations of the number 15 as the sum of three distinct single-digit positive integers. On every move, each player must write a sum that has exactly one common addend with the previous sum, no common addends with the second previous sum, and less than three common addends with any sum written earlier. Ats starts and can choose the first sum arbitrarily. The player who cannot write a sum loses. Which player can win regardless of his opponent's play?

Answer: Pets.

Solution: There are 8 possible sums:

$$\begin{array}{lll} 1+5+9, & 2+6+7, & 3+4+8, \\ 1+6+8, & 2+4+9, & 3+5+7, \\ 2+5+8, & 4+5+6. & \end{array}$$

Note that no two of the three sums in the first row have any common addends, and the same applies to the three sums in the second row. Moreover, each sum in the first two rows has one common addend with every sum outside its own row, including the sums in the third row. Thus, Pets can choose one of the first two rows (one from which Ats did not choose a sum in the first move) and write any sum from it on his turn. This guarantees a common addend with the sum that Ats has written on his turn. Since Ats cannot write a sum from the same row from which Pets has chosen his sum on his second move, Pets can choose any unused sum from that row on his second move, because it has a common addend with the sum written by Ats and no common addends with the sum last written by Pets. Analogously, Pets can make his third move if Ats has not already lost by then.

When Pets has made his third move, there are only two sums left unused. This means that if Ats were to make another move, one of the sums in the third row should have been written on the board either on this move or on an earlier move. According to Pets' strategy, Ats had to do this. However, the sums in the third row have one common addend with every other sum, so according to the rules they cannot have either the second previous sum or the sum after the next sum. This contradicts the assumption that the game can continue after Pets' third move. So in fact, Ats has no more suitable sums left after Pets' third move and has lost.

F15 *(Grade 9.)*

(a) Every side and diagonal of a regular 2025-gon is coloured either red or blue. Can it happen that the same number of red and blue line segments meet at each vertex?

(b) The same question if only the diagonals are coloured.

Answer: (a) Yes; (b) No.

Solution 1:

(a) Each side and diagonal connects vertices that are 1 to 1012 side lengths apart as you move along the polygon. Exactly 2 sides or diagonals of a given length meet at each vertex. We color the sides and diagonals that have an odd number of sides between their endpoints red and all other diagonals blue. Since there are an equal number of odd and even lengths, there are also an equal number of 1012 red and 1012 blue segments meeting at each vertex.

(b) Suppose we can color the diagonals as required. Then each vertex has 1011 red (and 1011 blue) diagonals. Adding up the numbers of red diagonals that meet at each vertex, we get $1011 \cdot 2025$, which is an odd number. On the other hand, each diagonal has been counted exactly 2 times (once at each of its endpoints), so the sum should be even. The contradiction shows that the situation is not possible.

Solution 2: Consider a graph whose vertices are the vertices of a given polygon and whose edges are its colored sides and diagonals.

(a) We note that each vertex has 2024 adjacent edges, which is an even number. Hence there is an Euler cycle in this graph. In total, there are $\frac{2025 \cdot 2024}{2} = 2025 \cdot 1012$ edges in this graph, which is also an even number. We can color the edges in the resulting Euler cycle alternately red and blue. Then there will be one red and one blue edge at each vertex when we pass through that vertex. Since we eventually pass through all the edges and return to the starting vertex, the numbers of blue and red edges at each vertex will be equal.

(b) Now there are $\frac{2025 \cdot 2022}{2} = 2025 \cdot 1011$ edges in total, which is odd. If the same number of red and blue line segments meet at each vertex then the total numbers of red and blue segments must be the same. So the total number of edges should be even, a contradiction.

F16 *(Grade 10.)* Anna, Berta and Carol make fruit drinks from syrup. Anna makes a litres of drink by mixing water and syrup in the proportion of $a : 1$. Berta makes b litres of drink by mixing water and syrup in the proportion of $b : 1$. Carol makes c litres of drink by mixing water and syrup in the proportion of $c : 1$. (It is not known if a, b and c are integers.) They make 6 litres of drink in total. Prove that they use at most 2 litres of syrup.

Solution 1: The percentage of syrup in Anna's drink is $\frac{1}{a+1}$, thus it contains $\frac{a}{a+1}$ litres of syrup. Similarly, Berta uses $\frac{b}{b+1}$ litres and Carol uses $\frac{c}{c+1}$ litres of syrup. Hence we have to prove that $a+b+c=6$ implies

$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}\leqslant 2.$$

As $\frac{a}{a+1}=1-\frac{1}{a+1}$ and similarly $\frac{b}{b+1}=1-\frac{1}{b+1}$ and $\frac{c}{c+1}=1-\frac{1}{c+1}$, this is equivalent to the inequality

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\geqslant 1. \qquad (12)$$

Multiplying both sides by 3 leads to the equivalent inequality

$$\frac{3}{a+1}+\frac{3}{b+1}+\frac{3}{c+1}\geqslant 3.$$

Adding 3 to both sides and applying $3=\frac{6+3}{3}=\frac{a+b+c+3}{3}=\frac{a+1}{3}+\frac{b+1}{3}+\frac{c+1}{3}$ in the left hand side, we obtain the equivalent inequality

$$\frac{3}{a+1}+\frac{a+1}{3}+\frac{3}{b+1}+\frac{b+1}{3}+\frac{3}{c+1}+\frac{c+1}{3}\geqslant 6. \qquad (13)$$

The sum of every positive real number and its reciprocal is at least 2. Hence the inequality (13) holds for every a, b and c.

Solution 2: As in Solution 1, we reduce the problem to the inequality (12). After converting the fractions to a common denominator, removing parentheses and collecting similar terms, it suffices to show that

$$\frac{ab+bc+ca+2(a+b+c)+3}{abc+ab+bc+ca+a+b+c+1}\geqslant 1,$$

or equivalently,

$$ab+bc+ca+2(a+b+c)+3\geqslant abc+ab+bc+ca+a+b+c+1.$$

Applying the assumption $a+b+c=6$ and collecting similar terms reduces this inequality to $8\geqslant abc$. By AM-GM, $2=\frac{6}{3}=\frac{a+b+c}{3}\geqslant\sqrt[3]{abc}$. Hence $8=2^3\geqslant abc$, completing the proof.

Solution 3: As in Solution 1, we reduce the problem to the inequality (12). The latter is equivalent to the inequality

$$\frac{\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}}{3}\geqslant\frac{1}{3},$$

which in turn is equivalent to the inequality

$$\frac{3}{\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}}\leqslant 3.$$

By AM-HM,

$$\frac{3}{\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}} \leqslant \frac{(a+1)+(b+1)+(c+1)}{3} = \frac{a+b+c+3}{3}.$$

As $a+b+c=6$, we obtain $\frac{a+b+c+3}{3}=3$, completing the proof.

F17 *(Grade 10.)* The incentre of a triangle ABC is I. Points D and E on the sides AB and AC, respectively, satisfy $DI \perp BI$ and $EI \perp CI$. Prove that the line DE is tangent to the incircle of the triangle ABC.

Solution 1: Let X and Y be the reflections of points D and E, respectively, across the point I (Fig. 32). Then $\angle XBI = \angle IBD = \angle IBA = \angle CBI$, implying that X lies on the line BC. As $\angle DIB = 90^\circ$, points D, I and X lie on a line, i.e., X is the point of intersection of lines ID and BC. Analogously, we see that Y is the point of intersection of lines IE and BC. Hence $\angle EID = \angle YIX$, which along with the equalities $IX = ID$ and $IY = IE$ shows that triangles DEI and XYI are equal.

Thus also the altitudes drawn from the vertex I in triangles DEI and XYI are equal. The altitude drawn from the vertex I of the triangle XYI equals the inradius of the triangle ABC. Hence the same holds for the triangle DEI, i.e., the incircle of the triangle ABC passes through the foot of the altitude drawn from the vertex I of the triangle DEI. The line DE is perpendicular to this altitude which is the inradius of the triangle ABC; consequently, the line DE is tangent to the incircle of the triangle ABC.

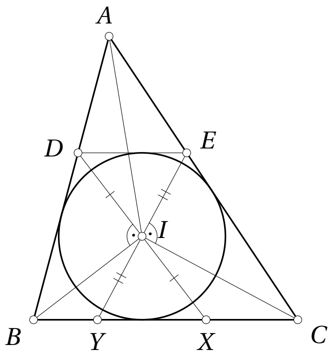


Fig. 32

Solution 2: Let U and V be the reflections of points B and C, respectively, across the point I (Fig. 33). Then $IU = IB$ and $IV = IC$, implying that $BC \parallel UV$. As the line BC is tangent to the incircle of the triangle ABC whose centre I is the centre of reflection, the line UV is also tangent to the incircle of the triangle ABC by symmetry.

Now let the line UV intersect the side AB and AC at points D' and E', respectively (Fig. 34). Then $\angle D'UB = \angle CBU = \angle CBI = \angle IBA = \angle UBD'$,

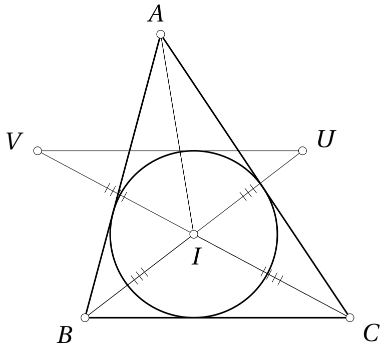


Fig. 33

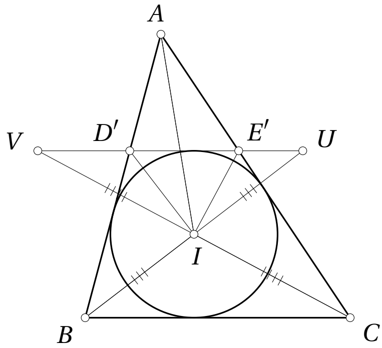


Fig. 34

implying $D'B = D'U$. As $D'I$ is the median drawn from the vertex angle of the isosceles triangle $D'BU$, it must also be its altitude. This implies $D'I \perp BI$ yielding $D' = D$.

Analogously, we get $E' = E$. Hence $DE = D'E' = UV$. Altogether, we have shown that the line DE is tangent to the incircle of the triangle ABC.

Solution 3: Denote $\angle BAI = \angle IAC = \alpha$, as well as $\angle CBI = \angle IBA = \beta$ and $\angle ACI = \angle ICB = \gamma$. Let K be the circumcentre of the triangle DEI (Fig. 35). By construction, $\alpha + \beta + \gamma = \frac{180^\circ}{2} = 90^\circ$.

From the triangle BCI, we get

$$\angle BIC = 180^\circ - (\beta + \gamma) = 180^\circ - (90^\circ - \alpha) = 90^\circ + \alpha,$$

hence $\angle EID = 360^\circ - 90^\circ - 90^\circ - (90^\circ + \alpha) = 90^\circ - \alpha$. From the circumcircle of the triangle DEI, we now get

$$\angle EKD = 2\,(90^\circ - \alpha) = 180^\circ - 2\alpha = 180^\circ - \angle DAE.$$

Thus the quadrilateral $ADKE$ is cyclic. Its chords DK and EK are equal, implying that the corresponding inscribed angles are also equal, i.e., $\angle DAK = \angle KAE$. Hence K lies on the bisector of the angle BAC, i.e., on the line AI.

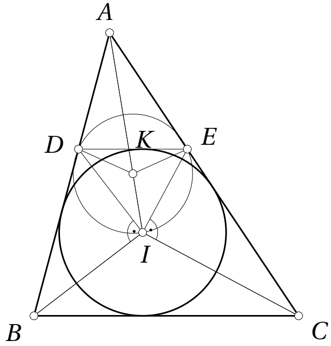


Fig. 35

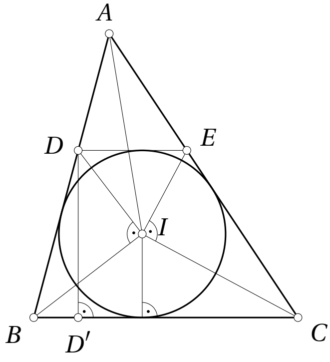


Fig. 36

From the triangle ABI, we get

$$\angle AIB = 180^\circ - (\alpha + \beta) = 180^\circ - (90^\circ - \gamma) = 90^\circ + \gamma,$$

hence $\angle KID = \angle AID = (90^\circ + \gamma) - 90^\circ = \gamma$. Therefore also $\angle IDK = \gamma$ as $DK = IK$. On the other hand, $\angle KDE = \frac{180^\circ - (180^\circ - 2\alpha)}{2} = \alpha$, implying $\angle IDE = \gamma + \alpha = 90^\circ - \beta$. Since also $\angle BDI = 90^\circ - \beta$, the line DI is the external bisector of the angle ADE, whereas I is the point of intersection of this external bisector and the internal bisector of DAE. This means that I is the excentre of the triangle ADE. As the incentre of the triangle ABC is I and the incircle of ABC is tangent to the prolongation of the side AD of the triangle ADE, these circles must coincide. Hence DE is tangent to the incircle of the triangle ABC.

Solution 4: Let r be the inradius of the triangle ABC. We show that the point D lies at distance $2r$ from the side BC.

To this end, let D' be the projection of the point D on the side BC (Fig. 36) and let $\angle CBI = \angle IBA = \beta$. Then $BI = \frac{r}{\sin\beta}$, $DB = \frac{BI}{\cos\beta} = \frac{r}{\sin\beta\cos\beta}$ and $DD' = DB\sin 2\beta = \frac{r}{\sin\beta\cos\beta} \cdot 2\sin\beta\cos\beta = 2r$.

Analogously, we can show that the point E lies at distance $2r$ from the side BC. Thus the line DE is parallel to the side BC and at distance $2r$ from it. As the line BC is tangent to the incircle of the triangle ABC, also the line DE is tangent to this circle.

F18 *(Grade 10.)* Let n be any natural number. Find the least natural number k such that it is possible to write a natural number from 1 through k into every cell of an $n \times n$ table in such a way that the sum of every two cells with a common side differs from all other such sums. The numbers in different cells do not have to be distinct.

Answer: $n^2 - n + 1$.

Solution: The least and the largest among the sums of two natural numbers from 1 through k are $1 + 1 = 2$ and $k + k = 2k$, respectively. Thus there can be at most $2k - 1$ distinct sums. The number of distinct pairs of cells with a common side is $n - 1$ in every row and column. As the total number of rows and columns is $2n$, the number of such pairs is $2n(n-1) = 2n^2 - 2n$. Hence $2k - 1 \geqslant 2n^2 - 2n$ which implies

$$k \geqslant \frac{2n^2 - 2n + 1}{2} = n^2 - n + \frac{1}{2}.$$

As k is an integer, we have $k \geqslant n^2 - n + 1$.

Now we show that $k = n^2 - n + 1$ is achievable. To this end, partition the $n \times n$ table into $2n - 1$ diagonals (directed from top right to bottom left; see Fig. 37). We fill the diagonals starting from the top left corner with consecutive integers $1, 2, \ldots$, but whenever switching from a diagonal with an odd number to the next diagonal with an even number, we repeat the number just written. This repetition happens $n - 1$ times. Hence the last

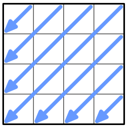


Fig. 37

1	1	3	5
2	4	6	9
5	7	10	11
8	11	12	13

Fig. 38

number written into the bottom right cell is $n^2 - n + 1$ as desired. Figure 38 depicts the situation for $n = 4, k = 13$.

We show that the sum of every two cells with a common side is unique. Each sum under consideration is obtained by adding numbers in cells of two consecutive diagonals. All sums of cells of the same two diagonals are obviously distinct, because when moving from the top right end to the bottom left end, the sum strictly increases at every step. When considering distinct pairs of consecutive diagonals, the sums must also be distinct. Indeed, let one pair under consideration contain diagonals No a and $a+1$ and let the other pair contain diagonals No b ja $b+1$, where $a < b$. Then numbers in the first diagonal of the first pair do not exceed numbers in the first diagonal of the second pair, whereas numbers in the second diagonal of the first pair do not exceed numbers in the second diagonal of the second pair. In either case, equality can hold only if $b = a + 1$. But the equality cannot hold for both cases simultaneously, because a and $a+1$ have distinct parities, implying that, by construction, when switching either from the diagonal No a to the diagonal No $a+1$ or from the diagonal No b to the diagonal No $b+1$, the last number is not repeated.

F19 *(Grade 10.)* A point E is chosen on the side AB of a rectangle $ABCD$ ($E \neq A$, $E \neq B$). The line segments BD and CE intersect at point F. Among the triangles ADE, DEF, DCF, BCF and BEF, there are exactly two pairs of triangles with equal area (the order of components in a pair is not taken into account). Find the ratio of the lengths of the line segments EB and AB.

Answer: $\frac{\sqrt{2}}{2}$.

Solution: Denote the area of a figure $\mathcal{K}$ by $S_{\mathcal{K}}$. As

$$S_{DEF} + S_{DCF} = S_{CDE} = \frac{1}{2} S_{ABCD} = S_{BCD} = S_{BCF} + S_{DCF},$$

we have $S_{DEF} = S_{BCF}$ (Fig. 39). Hence (DEF, BCF) is one pair of triangles with equal area regardless of the choice of point E. In order to have exactly

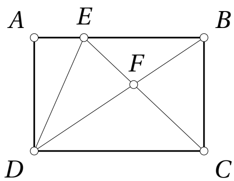


Fig. 39

two such pairs, none of the remaining three triangles can have the same area as triangles DEF and BCF. Thus the second pair must come from among triangles ADE, BEF and DCF. On the other hand,

$$S_{DEF} + S_{DCF} = S_{CDE} = \frac{1}{2} S_{ABCD} = S_{ABD} = S_{ADE} + S_{DEF} + S_{BEF},$$

implying $S_{DCF} = S_{ADE} + S_{BEF}$. Hence each of triangles ADE and BEF has an area smaller than that of the triangle DCF. Thus the second pair of triangles with equal area is (ADE, BEF). Taking into account the equality $S_{ADE} + S_{BEF} = S_{DCF}$, we obtain $S_{ADE} = S_{BEF} = \frac{1}{2} S_{DCF}$.
As $\angle EBF = \angle CDF$ and $\angle FEB = \angle FCD$, triangles BEF and DCF are similar. Hence $\frac{S_{BEF}}{S_{DCF}} = \left(\frac{EB}{CD}\right)^2 = \left(\frac{EB}{AB}\right)^2$, implying

$$\frac{EB}{AB} = \sqrt{\frac{S_{BEF}}{S_{DCF}}} = \sqrt{\frac{\frac{1}{2} S_{DCF}}{S_{DCF}}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}.$$

F20 *(Grade 11.)* Find the least positive integer n such that:
(a) both n and $n+1$ are divisible by the squares of two distinct prime numbers;
(b) both n and $n+3$ are divisible by the squares of two distinct prime numbers.
Answer: (a) 675; (b) 2025.
Solution: Both the pair $(n, n+1)$ and $(n, n+3)$ must contain one odd number. This odd number must be divisible by the square of one of the following numbers:

$$15 = 3 \cdot 5, 21 = 3 \cdot 7, 33 = 3 \cdot 11, 35 = 5 \cdot 7, 39 = 3 \cdot 13, 45 = 3^2 \cdot 5, \ldots.$$

Their squares are $225, 441, 1089, 1225, 1521, 2025, \ldots$. Thus either n or $n+1$ or $n+3$ must be a multiple of one of these squares.
(a) We cannot have $n+1 = 225$ or $n = 225$, as neither $224 = 2^5 \cdot 7$ or $226 = 2 \cdot 113$ is divisible by two prime squares. Similarly we cannot have $n+1 = 441$ or $n = 441$, as $440 = 2^3 \cdot 5 \cdot 11$ and $442 = 2 \cdot 13 \cdot 17$. We also cannot have $n+1 = 3 \cdot 225 = 675$, as $674 = 2 \cdot 337$. But $n = 675$ works, as $676 = 2^2 \cdot 13^2$. Thus the desired n is 675.
(b) In the solution to part (a) we saw that both 675 and 676 are divisible by two prime squares. Therefore, so must also the numbers $3 \cdot 675 = 2025$ and $3 \cdot 676 = 2028$. Thus $n = 2025$ has the desired properties. It remains to show that there are no smaller such numbers.
The only option for n or $n+3$ not divisible by 2^2 or 3^2 is 1225, but we cannot have either $n+3 = 1225$ or $n = 1225$, as $1222 = 2 \cdot 13 \cdot 47$ and $1228 = 2^2 \cdot 307$. We may now assume that one of n and $n+3$ is divisible by 2^2 and the other by 3^2, as they cannot be divisible by the same prime square.
The odd number out of n and $n+3$ is either one of the odd multiples of 225, which are 225, 675, 1125 and 1575, one of the odd multiples of 441, which

are 441 and 1323, or one of 1089 and 1521. For each of them, only one of the numbers differing from it by 3 is divisible by 2^2, so it remains to check that these numbers do not contain any other prime factors squared:

$$\begin{aligned}
225 \to\ 228 &= 2^2 \cdot 3 \cdot 19,\\
675 \to\ 672 &= 2^2 \cdot 3 \cdot 56,\\
1125 \to 1128 &= 2^2 \cdot 3 \cdot 94,\\
1575 \to 1572 &= 2^2 \cdot 3 \cdot 131,\\
441 \to\ 444 &= 2^2 \cdot 3 \cdot 37,\\
1323 \to 1320 &= 2^2 \cdot 3 \cdot 110,\\
1089 \to 1092 &= 2^2 \cdot 3 \cdot 91,\\
1521 \to 1524 &= 2^2 \cdot 3 \cdot 127.
\end{aligned}$$

F21 *(Grade 11.)* The coefficients of the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

are real numbers such that $a_i = a_{n-i}$ for every $i = 0, 1, \ldots, n$, and $a_n \neq 0$. Let $x_1, x_2, \ldots, x_k$ be all the real roots of the polynomial $P(x)$ without repetitions.

(a) Prove that

$$|x_1| + |x_2| + \ldots + |x_k| \geqslant k.$$

(b) Is the inequality definitely strict in the case $k > 1$?

Answer: (b) No.

Solution:

(a) Let c be any root of $P(x)$. As $P(0) = a_0 = a_n \neq 0$, we must have $c \neq 0$. Notice that

$$\begin{aligned}
P\left(\frac{1}{c}\right) &= a_n \left(\frac{1}{c}\right)^n + a_{n-1} \left(\frac{1}{c}\right)^{n-1} + \ldots + a_1 \left(\frac{1}{c}\right) + a_0\\
&= \left(\frac{1}{c}\right)^n \left(a_0 c^n + a_1 c^{n-1} + \ldots + a_{n-1} c + a_n\right)\\
&= \left(\frac{1}{c}\right)^n \left(a_n c^n + a_{n-1} c^{n-1} + \ldots + a_1 c + a_0\right)\\
&= \left(\frac{1}{c}\right)^n P(c) = 0,
\end{aligned}$$

which shows that $\frac{1}{c}$ is also a root of $P(x)$. Thus the roots can be divided into inverse pairs.

For each root c we have $|c| + \left|\frac{1}{c}\right| \geqslant 2$. The only roots paired with itself are 1 and -1, both of which have an absolute value of 1. If there are l such roots, then

$$|x_1| + |x_2| + \ldots + |x_k| \geqslant \frac{k-l}{2} \cdot 2 + l \cdot 1 = k - l + l = k.$$

(b) The polynomial $P(x) = x^3 - x^2 - x + 1 = (x-1)^2(x+1)$ satisfies the conditions of the problem and its real roots are $x_1 = 1$ and $x_2 = -1$. Thus $k = 2 > 1$, but $|x_1| + |x_2| = 2 = k$, so the inequality is non-strict.

F22 *(Grade 11.)* Circles ω_1 and ω_2 touch at point K. The line through the centres of the circles intersects the circle ω_1 once more at point A. A line

through the point A intersects the circle ω_1 once more at point B and the circle ω_2 at points C and D, where the points A, B, C, D lie on the line in the order. Given that the line segments AB, BC and CD have equal lengths, find the ratio of the radii of the circles ω_1 and ω_2.
Answer: $\frac{1}{3}$.
Solution 1: Let the radii of ω_1 and ω_2 be r_1 and r_2 respectively. Let E be the second intersection of AK and ω_2 (Fig. 40). As AK and KE are diameters of ω_1 and ω_2 respectively, the angles ABK and KDE must be right angles.
In triangle AKC, the segment KB is both a median and an altitude, thus $KA = KC$ and $\angle KAC = \angle ACK$. As $CDEK$ is cyclic, we have

$$\angle EAD = \angle KAC = \angle ACK = \angle DEA.$$

Thus ADE is isosceles, also KBA and KDE are similar. Therefore

$$\frac{r_1}{r_2} = \frac{AK}{KE} = \frac{AB}{DE} = \frac{AB}{AD} = \frac{1}{3}.$$

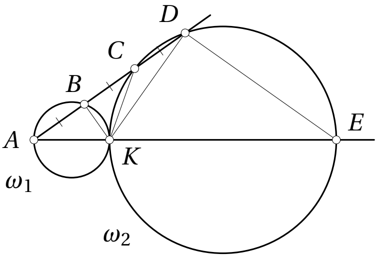


Fig. 40

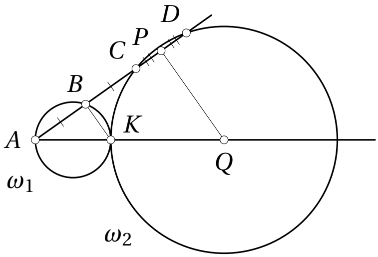


Fig. 41

Solution 2: Let the radii of ω_1 and ω_2 be r_1 and r_2 respectively. Let P be the midpoint of CD and Q the center of ω_2 (Fig. 41). Then PQ is perpendicular to CD, as it connects the midpoint of the chord CD of ω_2 and the center of ω_2. Thus $\angle APQ = \angle CPQ = 90^\circ$. As AK is a diameter of ω_1, we have $\angle ABK = 90^\circ$. Therefore $BK \parallel PQ$. Together with $AB = BC = CD$, we get $\frac{AQ}{AK} = \frac{AP}{AB} = 2.5$ or $2AQ = 5AK$. From here $2\left(2r_1 + r_2\right) = 5 \cdot 2r_1$, from which $\frac{r_1}{r_2} = \frac{1}{3}$.
Remark: The conditions of the problem also uniquely determine the angle between AK and AB. It can easily be shown that $\tan\angle KAB = \frac{\sqrt{2}}{2}$, from which $\angle KAB \approx 35.3^\circ$. Indeed, denoting $\angle KAB = \alpha$ and $\angle BDK = \delta$, we have $\tan\alpha = \frac{BK}{AB} = \frac{2BK}{2AB} = 2 \cdot \frac{BK}{BD} = 2\tan\delta$, whereas $\sin\delta = \frac{BK}{KD} = \frac{1}{3}$. Therefore $\tan^2\delta = \frac{\sin^2\delta}{1-\sin^2\delta} = \frac{\frac{1}{9}}{1-\frac{1}{9}} = \frac{1}{8}$, from which $\tan\delta = \frac{1}{2\sqrt{2}}$, yielding $\tan\alpha = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

F23 *(Grade 11.)* Find all positive integers n for which all 10^n non-negative integers consisting of n digits can be ordered in such a way that all the

following conditions are met:
(1) the first number consists of zeros only;
(2) every two numbers that are consecutive in this order differ at exactly one position and the digits at this position differ by exactly 1;
(3) the last number consists of nines only.

Remark: In this problem, we allow numbers to begin with zero(s).

Answer: All positive odd integers.

Solution 1: Assume that for some n, a suitable ordering exists. From condition 2 we see that the sums of digits of any two consecutive numbers differ by exactly 1, and thus have opposite parities. As the first number has an even sum of digits, the 10^n-th number must have an odd sum of digits. As the latter sum is $9n$, this means that the conditions can only be satisfied for odd n.

Now let n be odd. We prove by induction that a suitable ordering exists. For $n = 1$, the natural ordering works. Now let $n > 1$ and assume that an ordering exists for $n - 2$. We first order all numbers starting with 00, using the existing ordering for the other digits. Then we order the numbers starting with 01 in the reverse existing ordering for the other digits, then the numbers starting with 02 in the original order, and so on. During this process, we change the first two digits in the following order:

$$\begin{aligned}
&00 \to 01 \to \ldots \to 09 \to 19 \to 18 \to \ldots \to 10 \to \\
&\to 20 \to 21 \to \ldots \to 29 \to 39 \to 38 \to \ldots \to 30 \to \\
&\to 40 \to 41 \to \ldots \to 49 \to 59 \to 58 \to \ldots \to 50 \to \\
&\to 60 \to 61 \to \ldots \to 69 \to 79 \to 78 \to \ldots \to 70 \to \\
&\to 80 \to 81 \to \ldots \to 89.
\end{aligned}$$

The last number written so far ends with zeros. It remains to order the numbers starting with a 9. We divide them into groups based on their last $n - 2$ digits. We order these groups by the ordering existing for numbers with $n - 2$ digits. Within the first group, we order the numbers in the descending order, within the second group in the ascending order etc. There are an even number of groups, so the final group is in the ascending order. Thus the created ordering will satisfy the conditions of the problem.

Solution 2: We exclude the even n like in Solution 1.

Like in Solution 1, we find an ordering for odd n by induction. We again divide the numbers into groups based on their last $n - 2$ digits and order the groups by the existing ordering. There are again an even number of groups.

In odd-numbered groups, we order the numbers in the group based on their first two digits, using the ordering in Fig. 42, starting from 00 and ending with 54. In even-numbered groups, we use the same ordering backwards, except for the final group, where we use the ordering in Fig. 43, starting with 54 and ending with 99.

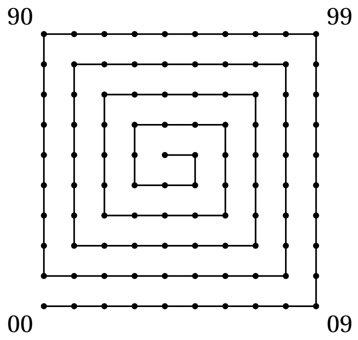


Fig. 42

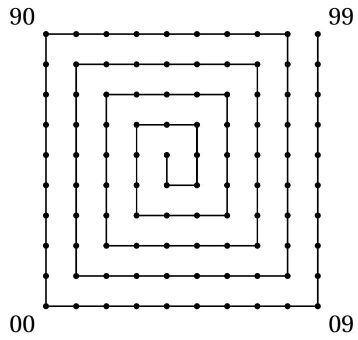


Fig. 43

F24 *(Grade 11.)* Every night, Juku listens to exactly 14 songs from a playlist containing exactly 100 songs. Every time a song ends, the next song is chosen from among all 100 songs with equal probability (the same song may also repeat). Prove that, on more than half of all nights, Juku listens some song more than once.

Solution: We show that the probability of listening to the same song multiple times is greater than 0.5, which is equivalent to the probability of listening to 14 distinct songs being less than 0.5. The latter probability can be expressed as

$$1 \cdot 0.99 \cdot 0.98 \cdot 0.97 \cdot 0.96 \cdot 0.95 \cdot 0.94 \cdot 0.93 \cdot 0.92 \cdot 0.91 \cdot 0.9 \cdot 0.89 \cdot 0.88 \cdot 0.87. \tag{14}$$

To prove the bound, notice that

$$\begin{aligned} 0.93 \cdot 0.87 &= (0.9+0.03)(0.9-0.03) = 0.9^2 - 0.03^2 < 0.9^2, \\ 0.92 \cdot 0.88 &= (0.9+0.02)(0.9-0.02) = 0.9^2 - 0.02^2 < 0.9^2, \\ 0.91 \cdot 0.89 &= (0.9+0.01)(0.9-0.01) = 0.9^2 - 0.01^2 < 0.9^2. \end{aligned}$$

Thus the product (14) is less than 0.9^7. We will now show that $0.9^7 < 0.5$. Using the property $0.9 \cdot \overline{0.ab} = \overline{0.ab} - \overline{0.0ab} \leqslant \overline{0.ab} - \overline{0.0a}$ repeatedly, we get

$$\begin{aligned} 0.9^7 &= 0.9 \cdot 0.9 \cdot 0.9 \cdot 0.9 \cdot 0.9 \cdot 0.9 \cdot 0.9 \\ &= 0.81 \cdot 0.9 \cdot 0.9 \cdot 0.9 \cdot 0.9 \cdot 0.9 \\ &\leqslant 0.73 \cdot 0.9 \cdot 0.9 \cdot 0.9 \cdot 0.9 \\ &\leqslant 0.66 \cdot 0.9 \cdot 0.9 \cdot 0.9 \\ &\leqslant 0.6 \cdot 0.9 \cdot 0.9 \\ &= 0.54 \cdot 0.9 \\ &\leqslant 0.49 < 0.5. \end{aligned}$$

This finishes the proof.

F25 *(Grade 12.)* Integers are assigned to variables x, y and z to satisfy the equation

$$x^3 + y^3 + z^3 - 3xyz = 2025.$$

Find all possible values of the sum $x + y + z$.

Answer: 3, 9, 27, 75, 225, 675.

Solution 1: The given equation is equivalent to the equation

$$x^3 + y^3 + z^3 = 2025 + 3xyz.$$

As $3 \mid 2025$ and $3 \mid 3xyz$, also $3 \mid x^3 + y^3 + z^3$. Thus $3 \mid x + y + z$ as an integer and its cube are congruent modulo 3. Let $x + y + z = 3m$.

W.l.o.g., let $x \leqslant y \leqslant z$ and $x = m - a$, $y = m + b$, $z = m + c$, where $a \geqslant 0$, $c \geqslant 0$ and $b = a - c$. Substituting for x, y, z in the given equation and simplifying leads to

$$3m\left(a^2 + b^2 + c^2 + ab + ac - bc\right) - a^3 + b^3 + c^3 + 3abc = 2025.$$

Substituting also $b = a - c$ now gives $3m\left(3a^2 - 3ac + 3c^2\right) = 2025$ or, equivalently,

$$m\left(a^2 - ac + c^2\right) = 225.$$

As $a^2 - ac + c^2 = (a - c)^2 + ac \geqslant 0$, the number m is a positive divisor of 225. The positive divisors of 225 are 1, 3, 5, 9, 15, 25, 45, 75, 225.

If $a = 0$ then $mc^2 = 225$; we obtain 1, 9, 25, 225 as possible values of m, giving 3, 27, 75, 675 as the corresponding values of $x + y + z$. If $a = 2c$ then $mc^2 = 75$; now m can be either 3 or 75, giving 9 and 225 as the corresponding values of $x + y + z$. In the remaining cases, $5 \mid m$ but $25 \nmid m$; then also $5 \mid a^2 - ac + c^2$ but $25 \nmid a^2 - ac + c^2$. Thus $5 \mid \left(a^2 - ac + c^2\right)(a + c) = a^3 + c^3$. Cubing integers that are incongruent modulo 5 produces results incongruent modulo 5. As $a^3 \equiv -c^3 \equiv (-c)^3 \pmod 5$, we must have $a \equiv -c \pmod 5$. Hence $0 \equiv a^2 - ac + c^2 \equiv (-c)^2 - (-c)c + c^2 = 3c^2 \pmod 5$, implying that $5 \mid c$. But then also $5 \mid a$ and $25 \mid a^2 - ac + c^2$. The contradiction shows that there are no other solutions.

Solution 2: It is easy to see that

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)\left(x^2 + y^2 + z^2 - xy - yz - zx\right). \qquad (15)$$

As $3 \mid 2025$, either $3 \mid x + y + z$ or $3 \mid x^2 + y^2 + z^2 - xy - yz - zx$. On the other hand, the representation

$$x^2 + y^2 + z^2 - xy - yz - zx = (x + y + z)^2 - 3(xy + yz + zx)$$

shows that $3 \mid x^2 + y^2 + z^2 - xy - yz - zx$ if and only if $3 \mid x + y + z$. Hence $3 \mid x + y + z$ anyway.

Let $x + y + z = 3m$. W.l.o.g., let $x \leqslant y \leqslant z$ and $x = m - a$, $y = m + b$, $z = m + c$, where $a \geqslant 0$, $c \geqslant 0$ and $a = b + c$. Using the equality (15) along with the equality

$$x^2 + y^2 + z^2 - xy - yz - zx = \frac{1}{2}\left((x - y)^2 + (y - z)^2 + (z - x)^2\right),$$

as well as the equations about x, y and z, the initial equation can be converted to

$$3m\left((a + b)^2 + (b - c)^2 + (a + c)^2\right) = 4050.$$

Substituting also $a = b + c$ gives $3m\left(6b^2 + 6bc + 6c^2\right) = 4050$ which reduces to
$$m\left(b^2 + bc + c^2\right) = 225.$$
Note that $b^2 + bc + c^2 = \frac{1}{2}\left((b+c)^2 + b^2 + c^2\right) \geqslant 0$; hence m is a positive divisor of 225. The positive divisors of 225 are 1, 3, 5, 9, 15, 25, 45, 75, 225.
If $b = 0$ then $mc^2 = 225$; we obtain 1, 9, 25, 225 as possible values of m. If $b = c$ then $mc^2 = 75$; now m can be either 3 or 75. It remains to check the possibilities $m = 5$, $m = 15$ and $m = 45$. In these cases $b^2 + bc + c^2 = d$ where d is 45, 15 or 5, respectively. As $b^2 + bc + c^2 = d$ is equivalent to $3b^2 + (b + 2c)^2 = 4d$, the number $4d - 3b^2$ is a perfect square. If $d = 5$ then $20 - 3b^2$ must be a perfect square which is impossible. If $d = 15$ then we have $3 \mid 4d - 3b^2$, hence $60 - 3b^2 = (3k)^2$, implying $20 - 3k^2 = b^2$ for some integer k which is impossible. If $d = 45$ then $3 \mid 4d - 3b^2$ again, hence $180 - 3b^2 = (3k)^2$, implying that $60 - 3k^2 = b^2$ for some integer k which is impossible.

F26 *(Grade 12.)* Find all functions f from the set of all non-negative real numbers to the set of all real numbers such that $f(1) = 1$ and
$$(f(x+y))^2 \leqslant f\left(x^2 - 2xy + y^2\right)$$
for all real numbers x and y that satisfy the inequality $x + y \geqslant 0$.
Answer: $f(x) = 1$.
Solution: The given inequality can be rewritten as
$$(f(x+y))^2 \leqslant f\left((x-y)^2\right).$$
Take $y = 1 - x$. Then $1 = 1^2 = (f(1))^2 \leqslant f\left((2x-1)^2\right)$. As $(2x-1)^2$ obtains all non-negative real values, we can conclude that $1 \leqslant f(z)$ for every non-negative real number z.
Now take $x \geqslant -\frac{1}{2}$ and $y = 1 + x$. Then $(f(2x+1))^2 \leqslant f(1) = 1$. This implies $f(2x+1) \leqslant 1$ because all values of f are non-negative by the above. As $2x+1$ obtains all non-negative real values, we can deduce that $f(z) \leqslant 1$ for every non-negative real number z.
Consequently, $f(z) = 1$ for every non-negative real number z. This function satisfies all conditions of the problem.

F27 *(Grade 12.)* In an acute triangle ABC with $AB < AC$, points D, E and F are the feet of the altitudes drawn from vertices A, B and C, respectively. Let the orthocenter of ABC be H and the midpoint of the side BC be M. Point K on the prolongation of the line segment EM beyond M and point L on the line segment FM satisfy $MK = ML = MD$. Prove that points K, L and H lie on a line.
Solution 1: Firstly, we prove that $BDLF$ is an isosceles trapezium (Fig. 44). As $\angle CEB = 90^\circ = \angle CFB$, points B, C, E and F lie on a circle with diameter BC. Hence M is the center of this circle. Consequently, $MB = MF$, implying

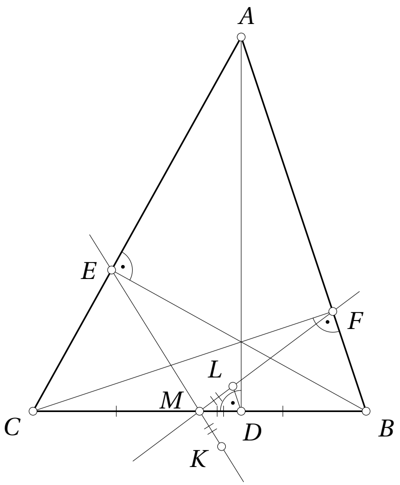


Fig. 44

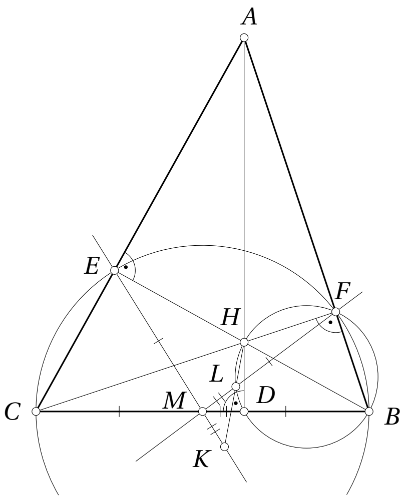


Fig. 45

$DB = MB - MD = MF - ML = LF$. Hence $DL \parallel BF$ and the desired claim follows.

Consequently, points B, D, L and F also lie on a circle (Fig. 45). Since $\angle BDH = 90^\circ = \angle HFB$, we know that BH is a diameter of this circle. Thus H lies on this circle, too. Now

$$\begin{aligned}\angle MLK &= \tfrac{\angle 180^\circ - \angle KML}{2} = \tfrac{\angle LME}{2} = \tfrac{\angle FME}{2} \\ &= \angle FBE = \angle FBH = \angle FLH.\end{aligned}$$

But $\angle MLK = \angle FLH$ implies that points K, L, and H lie on a line.

Solution 2: As in Solution 1, we prove that $BDLF$ is an isosceles trapezium whose circumcircle passes through H. By interchanging the roles of B and C, the roles of E and F, and the roles of K and L, we analogously obtain that $CKDE$ is an isosceles trapezium whose circumcircle passes through H (Fig. 46). Now $\angle LHB = \angle LFB = \angle FBD = \angle ABC$, but, on the other hand,

$$\angle KHB = 180^\circ - \angle EHK = \angle KCE = \angle CED.$$

As $\angle BDA = 90^\circ = \angle BEA$, the quadrilateral $ABDE$ is cyclic, implying that $\angle CED = \angle ABC$. Consequently, $\angle LHB = \angle KHB$, implying that points K, L and H lie on a line.

Solution 3: As in Solution 1, we prove that points B, D, L, F and H lie on a circle with diameter BH. By interchanging the roles of B and C, the roles of E and F, and the roles of K and L, we analogously obtain that points C, D, K, E and H lie on a circle with diameter CH. As $HL \perp BL$ and $HK \perp CK$ by Thales' theorem (Fig. 47), it suffices to prove that $BL \parallel CK$. To this end, note that

$$\begin{aligned}\angle LBM &= \angle LBD = \angle LFD = \angle MFD, \\ \angle KCM &= \angle KCD = \angle KED = \angle MED.\end{aligned}$$

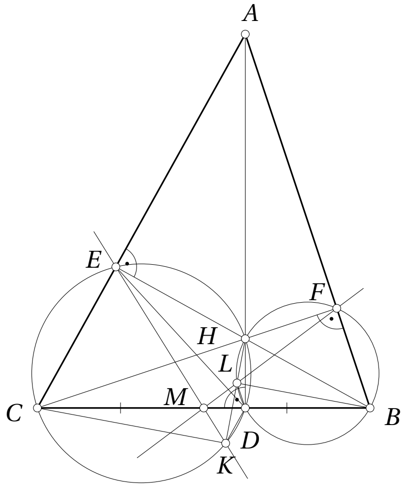


Fig. 46

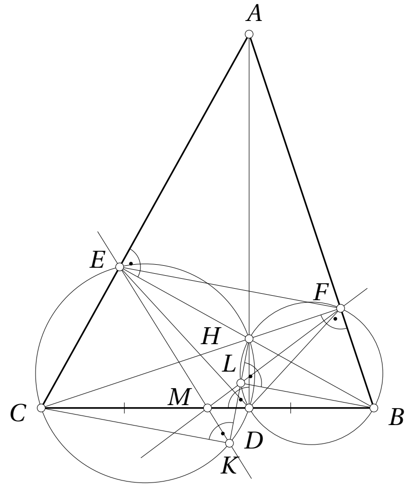


Fig. 47

It is known that the midpoints of sides of a triangle and the feet of altitudes of the triangle lie on a circle (so-called nine-point circle). Hence M, D, E and F are concyclic, implying that $\angle MFD = \angle MED$. Consequently, $\angle LBM = \angle KCM$, proving the desired result.

Remark: The claim used in Solution 3 that points D, E, F and M are concyclic can be easily proved. Denote $\angle CAB = \alpha$. In the cyclic quadrilateral $BCEF$, we get

$$\angle FME = 2\angle FBE = 2\angle ABE = 2\left(90^\circ - \alpha\right) = 180^\circ - 2\alpha.$$

On the other hand, using cyclic quadrilaterals $CAFD$ and $ABDE$, we obtain

$$\angle FDE = 180^\circ - \angle BDF - \angle EDC = 180^\circ - \alpha - \alpha = 180^\circ - 2\alpha.$$

Hence $\angle FME = \angle FDE$, implying the desired claim.

F28 *(Grade 12.)* On a plane, 5 points are chosen arbitrarily. Find the largest possible number of distinct right triangles with all vertices in the chosen points.

Answer: 8.

Solution 1: A square $ABCD$ and its centre E determine 8 distinct right triangles ABC, BCD, CDA, DAB, AEB, BEC, CED, DEA (Fig. 48).

We show that more than 8 right triangles is impossible. Firstly, note that among any 5 points, one can choose 4 points that are vertices of a rectangle in at most one way. Indeed, suppose that this can be done in two different ways. Then these two quadruples of points have 3 points in common. But 3 vertices of a rectangle (even parallelogram) uniquely determine the last vertex. Secondly, note that a quadrangle whose every three vertices form a right triangle is a rectangle. Indeed, these four triangles must have right angle at distinct vertices, otherwise three points would lie on a line. Now if A,

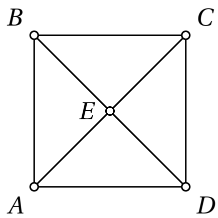


Fig. 48

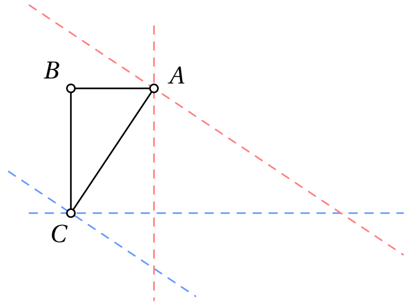


Fig. 49

B, C are the vertices of a right triangle with right angle at B then the remaining vertex D must be located in such a way that some triangle could have right triangle at A and some triangle could have right angle at C. Hence D must lie on a line passing through A and perpendicular to either AB or AC, and also on a line passing through C and perpendicular to either AC or BC (in Fig. 49, these lines are drawn using dashes of distinct colours). As lines perpendicular to AC to not meet, D must lie on the line passing through A perpendicular to AB or on the line passing through C perpendicular to BC. If D lies on both these lines, $ABCD$ is a rectangle. Hence consider the case with D lying on exactly one of these lines. W.l.o.g., let D lie on the line passing through A perpendicular to AB and on the line passing through C perpendicular to AC. But then BCD is not a right triangle. Hence the only possibility for D is such that $ABCD$ is a rectangle.

Let now 5 points be chosen on a plane. Let us count right triangles with vertices in the chosen points by quadrangles, among the vertices of which the vertices of the triangle are. As there are 5 possibilities to extract 4 points out of the given 5 points and, by the above, at most one of the obtained quadruples enable 4 right triangles and the others consequently enable at most 3 right triangles, we can have at most $4 + 4 \cdot 3 = 16$ right triangles. But each of these right triangles is counted twice as the fourth vertex can be either of the two remaining points. Hence there are at most 8 distinct right triangles.

Solution 2: As in Solution 1, we show that it is possible to find 8 right triangles.

Now we prove that more than 8 right triangles is impossible. Suppose the opposite, i.e., there can be 9 right triangles. As there are $C_5^3 = 10$ triples of chosen points in total, at most one triple does not determine a right triangle. Let A and B be chosen points at maximal distance from each other (if there are several pairs of points with this property, take any of them). By choice, AB can only be the hypotenuse of a right triangle. Hence at least two of the remaining three chosen points must lie on the circle with diameter AB; let these points be C and D. At least one of ACD and BCD must be a right triangle. As all vertices of this triangle lie on the circle with diameter AB, two vertices of this triangle must be the endpoints of another diameter of this circle. These points can only be C and D. Therefore also CD is a line segment of maximal length and can only be the hypotenuse of a right triangle.

If the last chosen point E also lies on the circle with diameter AB then ACE and BCE are not right triangles because no pair of vertices can be the endpoints of a diameter of this circle. But if E does not lie on this circle then AEB and CED are not right angles, implying that ABE and CDE cannot be right triangles. The contradiction shows that there cannot be 9 right triangles.

F29 *(Grade 12.)* Does there exist a geometric progression, among the members of which there are

(a) 3, 45 and 2025;

(b) 3, $\frac{45}{\sqrt[3]{5}}$ and 2025?

Answer: (a) No; (b) Yes.

Solution:

(a) Let the common ratio of the progression be q. W.l.o.g., assume that $q > 1$. One can also assume that the first term of the progression is 3. Then $45 = 3 \cdot q^k$ and $2025 = 3 \cdot q^l$, where k and l are integers. This implies $q^k = 15$ and $q^l = 675$. Therefore $q^{kl} = 15^l$, as well as $q^{kl} = 675^k$. Since $15 = 3 \cdot 5$ and $675 = 3^3 \cdot 5^2$, the equality $15^l = 675^k$ implies $3^l \cdot 5^l = 3^{3k} \cdot 5^{2k}$ which simplifies to $5^{l-2k} = 3^{3k-l}$. As k and l are integers, this is possible only if $l - 2k = 3k - l = 0$. But then $k = (l - 2k) + (3k - l) = 0$ which is obviously false.

(b) If $q = \frac{15}{\sqrt[3]{5}}$ then the next term after 3 is $\frac{45}{\sqrt[3]{5}}$ and the term after the next term is $\frac{3 \cdot 15^3}{5} = 3^4 \cdot 5^2 = 2025$. Hence there exists a suitable geometric progression.

Selected Problems from the IMO Team Selection Contests

S1 Let $n \geqslant 3$ be any natural number. A real number is written into every vertex of a regular n-gon in such a way that numbers in any two neighbouring vertices differ by at most 1. Find the least non-negative real number C such that, regardless of the choice of the numbers in the vertices, there exist two neighbouring vertices in which numbers differ by at most C.

Answer: $\begin{cases} 1 & \text{if } n \text{ is even;} \\ \frac{\lfloor \frac{n}{2} \rfloor}{\lceil \frac{n}{2} \rceil}, & \text{if } n \text{ is odd.} \end{cases}$

Solution: Denote the vertices of the polygon as $V_1, \ldots, V_n$ and let the real numbers in these vertices be $v_1, v_2, \ldots, v_n$, respectively. Moreover, denote $V_{n+1} = V_1$ and $v_{n+1} = v_1$. Define the value of the side V_iV_{i+1} of the polygon to be $s_i = v_{i+1} - v_i$.

Observe that $v_1, \ldots, v_n$ satisfy the conditions of the problem if and only if the corresponding differences $s_1, \ldots, s_n$ satisfy $\max\{|s_1|, \ldots, |s_n|\} \leqslant 1$ and $s_1 + \ldots + s_n = 0$. Therefore the problem can be reformulated as finding the least non-negative real number C such that, for arbitrary real numbers $s_1, \ldots, s_n$, assumptions $\max\{|s_1|, \ldots, |s_n|\} \leqslant 1$ and $s_1 + \ldots + s_n = 0$ would imply $\min\{|s_1|, \ldots, |s_n|\} \leqslant C$.

Clearly $\min\{|s_1|, \ldots, |s_n|\} \leqslant 1$ for any choice of $s_1, \ldots, s_n$ satisfying the assumptions. If $n = 2k$ for some $k \in \mathbb{N}$ then choosing $s_1 = \ldots = s_k = 1$ and $s_{k+1} = \ldots = s_{2k} = -1$ would imply $\min\{|s_1|, \ldots, |s_n|\} = 1$. Hence $C = 1$.

Let now be $n = 2k + 1$ for some $k \in \mathbb{N}$. Taking $s_1 = s_2 = \ldots = s_{k+1} = \frac{k}{k+1}$ and $s_{k+2} = \ldots = s_{2k+1} = -1$ establishes $\min\{|s_1|, \ldots, |s_n|\} = \frac{k}{k+1}$. We show that $\min\{|s_1|, \ldots, |s_n|\} \leqslant \frac{k}{k+1}$ whenever $s_1, \ldots, s_n$ satisfy the assumptions. To this end, suppose the contrary, i.e. $\min\{|s_1|, \ldots, |s_n|\} > \frac{k}{k+1}$. Reorder $s_1, \ldots, s_n$ as $d_1, \ldots, d_n$ so that $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$. Note that there exists $l \in \{1, \ldots, n\}$ such that $d_l > \frac{k}{k+1}$ and $d_{l+1} < -\frac{k}{k+1}$. If $l \geqslant k+1$ then

$$\begin{aligned} 0 &= s_1 + \ldots + s_n = d_1 + \ldots + d_n \geqslant l \cdot d_l + (n-l)d_n \\ &> l \cdot \tfrac{k}{k+1} - (n-l) \geqslant (k+1) \cdot \tfrac{k}{k+1} - k = k - k = 0, \end{aligned}$$

contradiction. In the case $l < k+1$, the proof is analogous (one can consider the opposite differences $-d_n, -d_{n-1} \ldots, -d_1$). Hence $C = \frac{k}{k+1} = \frac{\lfloor \frac{n}{2} \rfloor}{\lceil \frac{n}{2} \rceil}$.

S2 Kati writes the numbers

$$2^0, 2^1, 2^2, \ldots, 2^{100}, 3^0, 3^1, 3^2, \ldots, 3^{100}, 6^0, 6^1, 6^2, \ldots, 6^{100}$$

on the board. In each step, she performs one of the following operations:

(1) She can pick two numbers and replace them with their greatest common divisor and least common multiple; or

(2) She can pick two numbers, one of which is divisible by the other, and replace them with some two numbers whose greatest common divisor and least common multiple would be the two picked numbers.

Find the least and the greatest sum of all numbers on the board that can be achieved via finitely many steps.

Answer: The greatest possible sum is $101 + 2 \cdot \frac{6^{101}-1}{5}$ and the least possible sum is $2^{102} - 3 \cdot 2^{51} + 2 \cdot 3^{50} + 3^{101}$.

Solution: At all points throughout the process, every number on the board can be written as $2^\alpha 3^\beta$ for some $0 \leqslant \alpha, \beta \leqslant 100$. Moreover, the list of exponents of both 2 and 3 does not change throughout the process. Indeed, in a step of the first kind, two numbers $2^{\alpha_1}3^{\beta_1}$ and $2^{\alpha_2}3^{\beta_2}$ are turned into $2^{\min(\alpha_1,\alpha_2)}3^{\min(\beta_1,\beta_2)}$ and $2^{\max(\alpha_1,\alpha_2)}3^{\max(\beta_1,\beta_2)}$, preserving both the exponents α_1, α_2 of 2 and the exponents β_1, β_2 of 3. A step of the second kind is just the reverse of a step of the first kind, so the exponents are preserved again.

Thus the lists of both the exponents of 2 and 3 are

$$\underbrace{0,\ldots,0}_{103 \text{ copies}},1,1,2,2,...,100,100.$$

By the rearrangement inequality, we obtain that the sum of the numbers on the board is the greatest when they are

$$\underbrace{1,\ldots,1}_{103 \text{ copies}},6,6,6^2,6^2,\ldots,6^{100},6^{100}.$$

The sum of these numbers equals $101 + 2 \cdot \frac{6^{101}-1}{5}$. These numbers can be achieved by performing a step of the first kind on the numbers 2^γ and 3^γ for every $\gamma = 1, \ldots, 100$.

By the rearrangement inequality, we obtain that the sum of the numbers on the board is the least when the lists of the exponents of 2 and 3 are sorted in opposite directions. This means that the numbers would be

$$\begin{array}{cccccccccc} 2^{100} & 2^{100} & 2^{99} & 2^{99} & 2^{98} & 2^{98} & \ldots\ldots & 2^{51} & 2^{51} & 2^{50} \\ 2^{50} & 2^{49} & 2^{49}\cdot 3 & 2^{48}\cdot 3 & 2^{48}\cdot 3^2 & 2^{47}\cdot 3^2 & \ldots\ldots & 2\cdot 3^{49} & 3^{49} & 3^{50} \\ 3^{50} & 3^{51} & 3^{51} & 3^{52} & 3^{52} & 3^{53} & \ldots\ldots & 3^{99} & 3^{100} & 3^{100}. \end{array}$$

Using the formula for the sum of a geometric series for each row, separately for numbers at odd positions and even positions, yields:

- for the first row, $2^{101} - 2^{50}$ and $2^{101} - 2^{51}$;
- for the second row, $3^{51} - 2^{51}$ and $3^{50} - 2^{50}$;
- for the third row, $\frac{1}{2}\left(3^{101} - 3^{50}\right)$ and $\frac{1}{2}\left(3^{101} - 3^{51}\right)$.

Thus the sum of all the numbers is $2^{102} - 3 \cdot 2^{51} + 2 \cdot 3^{50} + 3^{101}$. This is indeed achievable. For this, we will pair the first desired number with the last one, the second first number with the second last one, etc. The least common multiples of the pairs are $6^{100}, 6^{100}, \ldots, 6^{26}, 6^{26}, 6^{25}$ and the greatest common divisors are $\underbrace{1,1,\ldots,1}_{103 \text{ copies}},6,6,6^2,6^2,\ldots,6^{24},6^{24}$. Finally, 6^{25} is left over. This means that we can use steps of the second kind to achieve the desired numbers from the situation with the maximal sum.

S3 The angle bisectors of an acute triangle ABC meet at point I. The line AI meets the circumcircle of the triangle ABC at point D ($D \neq A$) and the side BC at point E. The line BI meets the circumcircle of the triangle CDI at point K whereas the line CI meets the circumcircle of the triangle BDI at point L ($K \neq I$, $L \neq I$).

(a) Prove that the line DI is tangent to the circumcircle of the triangle IKL.

(b) Prove that points A, K, L, E are concyclic.

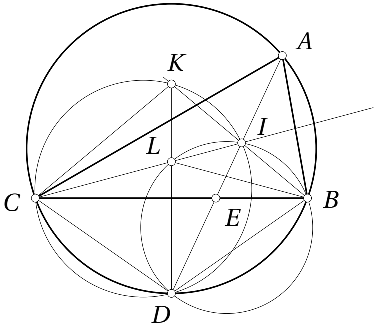


Fig. 50

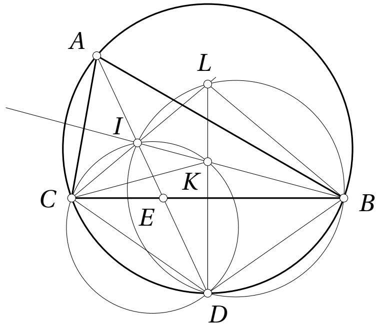


Fig. 51

Solution: Let $\alpha = \angle CAI = \angle IAB$, $\beta = \angle ABI = \angle IBC$, $\gamma = \angle BCI = \angle ICA$. Then $\alpha + \beta + \gamma = 90^\circ$ and $\angle CBD = \angle CAD = \alpha = \angle DAB = \angle DCB$, yielding

$$\angle KBD = \angle IBD = \alpha + \beta = 90^\circ - \gamma,$$
$$\angle DCL = \angle DCI = \alpha + \gamma = 90^\circ - \beta.$$

Depending on the location of the point K (Figures 50 and 51), we have either $\angle DKB = \angle DKI = \angle DCI$ or $\angle DKB = 180^\circ - \angle IKD = \angle DCI$; in each case $\angle DKB = 90^\circ - \beta$. Analogously, we obtain $\angle CLD = 90^\circ - \gamma$. Hence

$$\angle BDK = 180^\circ - (90^\circ - \gamma) - (90^\circ - \beta) = \beta + \gamma = 90^\circ - \alpha,$$
$$\angle LDC = 180^\circ - (90^\circ - \beta) - (90^\circ - \gamma) = \beta + \gamma = 90^\circ - \alpha.$$

On the other hand, we have $\angle BDC = 180^\circ - 2\alpha = 2\,(90^\circ - \alpha)$, meaning that both DK and DL bisect the angle BDC. Consequently, points D, K and L lie on a line. As the bisector of the vertex angle of the isosceles triangle BCD is also the perpendicular bisector of the line segment BC, symmetry yields $\angle DCK = \angle KBD = 90^\circ - \gamma$ and $\angle LBD = \angle DCL = 90^\circ - \beta$.

(a) If the point K lies between points C and I then

$$\angle DKI = \angle DKB = 90^\circ - \beta = \angle LBD = \angle LID.$$

Hence the line DI is tangent to the circumcircle of the triangle IKL (as points D, K, L lie on a line). If the point K lies between points I and D then

$$\angle ILD = \angle CLD = 90^\circ - \gamma = \angle DCK = \angle DIK.$$

Analogously to the previous case, the line DI must be tangent to the circumcircle of the triangle IKL.

(b) Since $\angle DKB = 90^\circ - \beta = \angle LBD$ and points D, K, L lie on a line, the line DB is tangent to the circumcircle of the triangle BKL. On the other hand, we have $\angle DAB = \alpha = \angle CBD = \angle EBD$, implying that DB is also tangent to the circumcircle of the triangle BEA. Using the power of the point D w.r.t. to these circles, we get $DB^2 = DK \cdot DL$ and $DB^2 = DA \cdot DE$, respectively. Altogether, we obtain $DA \cdot DE = DK \cdot DL$. Hence points A, K, L, E are concyclic.

S4 A triangle ABC with $AB < BC$ and an obtuse angle at vertex B is given. The incircle of the triangle ABC with incentre I touches the sides BC, CA and AB at points D, E and F, respectively. The line AI intersects the side BC at point K. The ray IB intersects the circumcircle of the triangle KIF at point P ($P \neq I$), whereas the ray IC intersects the circumcircle of the triangle KIE at point Q ($Q \neq I$). Prove that the line PQ bisects the line segment DK.

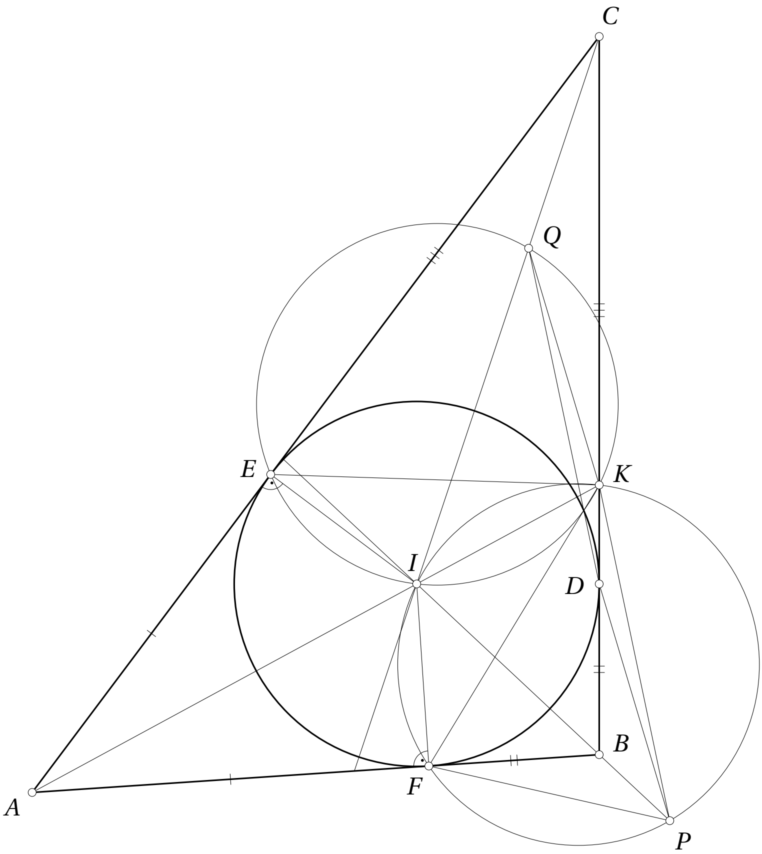


Fig. 52

Solution: Let $\angle BAC = 2\alpha$, $\angle CBA = 2\beta$, $\angle ACB = 2\gamma$; then $\alpha + \beta + \gamma = 90^\circ$. By the two tangents theorem, we have $AE = AF$ (Fig. 52), and because $\angle FAK = \alpha = \angle KAE$, the triangles AEK and AFK are equal. Thus also $KE = KF$. On the other hand, $\angle EIA = 90^\circ - \alpha = \angle AIF$, implying that $\angle KIE = 180^\circ - \angle EIA = 180^\circ - \angle AIF = \angle FIK$. This shows that the angles inscribed on equal chords KE and KF of the circumcircles of the triangles KIE and KIF are equal. Consequently, these circles have equal radii and

angles inscribed on equal chords of these circles are always equal.
From the triangle AIC, we obtain $\angle KIQ = \angle KIC = \alpha + \gamma = 90^\circ - \beta$ and $\angle FIP = \angle FIB = 90^\circ - \beta$. Thus angles inscribed on the chords KQ and PF of the circumcircles of the triangles KIE and KIF are equal, which implies $KQ = PF$. By the two tangents theorem, we also have $BF = BD$. As $\angle FBP = 180^\circ - \beta = \angle PBD$, the triangles FBP and DBP are equal. Hence also $PF = PD$. Altogether, we obtain $KQ = PD$.
Analogously, we get $PK = DQ$. Hence the quadrilateral $PDQK$ is a parallelogram and its diagonals PQ and DK bisect each other.
Remark: The proof works without the assumptions that $AB < BC$ and the angle at vertex B is obtuse, but it is harder to draw a picture.