



Estonian Math Competitions

2025/2026

University of Tartu Youth Academy
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Estonian Mathematical Olympiad

<https://www.math.olympiaadid.ut.ee/>

Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds: at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round.

In each round of the Olympiad, separate problem sets are given to the students of each grade from the 7th to the 12th. In the last five years, the final round was organized also to grades 7 and 8; previously, these two grades participated at school and regional levels only. About 25 students of each grade reach the final round. Some towns, regions and schools organize math olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in spring in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

In addition to the Olympiad, open contests take place in September and in December. In addition to students of Estonian middle and secondary schools, Estonian citizens who are studying abroad may also participate in these contests. The participants must have never enrolled in a university or other higher educational institution. The contestants compete in two categories: Juniors and Seniors. In the former category, only students up to the 10th grade may participate. Being successful in the open contests generally assumes knowledge outside the school curriculum.

Based on the results of all competitions during the year, about 20 IMO team candidates are selected. The IMO team selection contest for them is held in April or May in two rounds. Each round is an IMO-style two-day competition with 4.5 hours to solve 3 problems on both days. Some problems in our selection contest are at the level of difficulty of the IMO but easier problems are usually also included.

The problems of previous competitions can be downloaded at the Estonian Mathematical Olympiads website.

Problems Listed by Topic

Number theory: O3, O7, O9, O13, O15, O19, O21, F1, F4, F5, F9, F13, F17, F21

Algebra: O1, O4, O8, O10, O14, O16, O20, O22, F2, F8, F14, F18, F22, F24, S1, S2

Geometry: O5, O11, O17, O23, F3, F6, F10, F12, F15, F19, S3

Discrete mathematics: O2, O6, O12, O18, F7, F11, F16, F20, F23

Problems

Selected Problems from Open Contests

O1 (*Juniors.*) Find the number of pairs of positive integers (x, y) for which the equality $20x + 25y = 2025$ holds.

O2 (*Juniors.*) Call a positive integer n *interesting* if it satisfies all of the following conditions:

- (1) the product of the digits of n is 2025;
- (2) the sum of the digits of n is 25;
- (3) no two adjacent digits of the number n are equal;
- (4) the number n is divisible by 5;
- (5) the number n is greater than 2025^2 .

How many interesting positive integers are there?

O3 (*Juniors.*) Find all positive integers n which have (not necessarily distinct) positive divisors a and b such that $n = (a + 1)(b + 1)$.

O4 (*Juniors.*) Initially, the number 1 is written on the blackboard. In each step, the number x currently on the blackboard is replaced by the number $\frac{x}{2nx + x + 1}$, where n is the step number. Juku makes 44 such steps in a row. Find the number that remains on the blackboard at the end.

O5 (*Juniors.*) In an acute triangle ABC , the side AB is shorter than the side AC , and the internal angle bisector at vertex A intersects the altitudes BE and CF at points X and Y , respectively. Let Z be the midpoint of the segment XY . Prove that the points A, E, F , and Z lie on the same circle.

O6 (*Juniors.*) The squares of a grid are colored black and white in a chessboard pattern (i.e., any two squares sharing a side have opposite colors). Initially, there is a white queen in one corner square and a black queen in the opposite corner. In one move, a queen can move to any other square in the same row, column, or diagonal. If the opponent's queen is on the path of the moving queen, it is removed from the board, and the player who made the move wins the game. White moves first, and the players take turns. However, each player must move their queen horizontally or vertically on their first move, diagonally on their second move, horizontally or vertically again on their third move, and so on, alternating. Can either player win against any counterplay by their opponent, and if so, who, if:

- (a) the dimensions of the grid are 3×3 ?
- (b) the dimensions of the grid are 4×4 , and an extra rule applies stating that a queen can only move to squares of the color of its starting square?

O7 (*Juniors.*) Find the smallest positive integer n for which the number $2025n$ is the cube of some even number.

O8 (*Juniors.*) The cats of Fidgetville all start meowing at the same time. Once they start meowing, they always do so for 1 minute continuously and then rest for 9 minutes, and after 9 minutes of rest, they always start me-

owing again. The dogs of Fidgetville bark every time someone runs past them, and they also do so for 1 minute, running along with the passerby and stopping after the minute has passed to wait for the next runner.

Juku and Miku run the same circular track, each at a constant speed, starting from the same point. Juku runs clockwise, completing a lap in 30 minutes, while Miku runs counterclockwise, completing a lap in 60 minutes. Initially, all the dogs run with Juku for 1 minute. Juku and Miku stop running when, at the time of a meeting, the dogs are again at the exact same place as them. It is known that at this exact moment, the cats' minute of meowing ends. For how many minutes during Juku and Miku's run did the dogs bark at the same time as the cats were meowing?

O9 (*Juniors.*) Find all pairs of positive integers (n, m) for which the number $n! + m!$ is not divisible by any prime number greater than 3.

O10 (*Juniors.*) In the correct addition $A + BC + DEF = GHIJ$, different letters represent different digits. Find all possibilities for the value of $I + J$.

O11 (*Juniors.*) Let AD , BE , and CF be the altitudes of an acute triangle ABC . Point P is the reflection of point F across the line BC . Point Q is the reflection of point P across the line BE .

- (a) Prove that the points P , D , and E lie on the same line.
- (b) Prove that the points Q , F , and E lie on the same line.

O12 (*Juniors.*) On a 5×5 grid one has to place k identical white tokens and k identical black tokens so that every unit square contains at most one token and when the grid together with the tokens is rotated by 90° around the center of the grid, the arrangement of tokens remains exactly the same.

- (a) Find all positive integers k for which this is possible.
- (b) How many suitable arrangements are there for the smallest such k ?

O13 (*Seniors.*) For each positive integer n , denote by $\tau(n)$ the number of all its positive divisors, and let $\sigma(n)$ denote the sum of all its positive divisors. Find all positive integers n for which $n + \tau(n) = \sigma(n)$.

O14 (*Seniors.*) Solve the following system in real numbers:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 + 7x_7 = 1, \\ x_2 + 2x_3 + 3x_4 + 4x_5 + 5x_6 + 6x_7 + 7x_1 = 2, \\ x_3 + 2x_4 + 3x_5 + 4x_6 + 5x_7 + 6x_1 + 7x_2 = 3, \\ x_4 + 2x_5 + 3x_6 + 4x_7 + 5x_1 + 6x_2 + 7x_3 = 4, \\ x_5 + 2x_6 + 3x_7 + 4x_1 + 5x_2 + 6x_3 + 7x_4 = 5, \\ x_6 + 2x_7 + 3x_1 + 4x_2 + 5x_3 + 6x_4 + 7x_5 = 6, \\ x_7 + 2x_1 + 3x_2 + 4x_3 + 5x_4 + 6x_5 + 7x_6 = 7, \end{cases}$$

where the variables are $x_1, x_2, x_3, x_4, x_5, x_6, x_7$.

O15 (*Seniors.*) Can the number 2025 be expressed as the difference of the cubes of two integers?

O16 (*Seniors.*) Does there exist a function defined on all real numbers and taking only real values, which satisfies the equation

$$f(x) + f(f(y)) + f(f(f(x+y))) = x + y$$

for all real numbers x and y ?

O17 (*Seniors.*) In an acute triangle ABC , the side AB is shorter than the side AC , and the incenter is I . Let J be the reflection of point I across the line BC . The segment AJ intersects the circumcircles of triangles BIJ and CIJ at points D and E , respectively, with $D \neq J$ and $E \neq J$. A circle with center K is tangent to the side DE of triangle IDE and to the extensions of its other two sides. Prove that the points B, K, I , and C are concyclic.

O18 (*Seniors.*) On a 7×7 grid, the centers of some squares are marked such that there is no rectangle whose vertices are all marked points and whose sides are parallel to the grid lines. Find the greatest possible number of marked points.

O19 (*Seniors.*) Find all pairs of integers (a, b) for which

$$a^3 - a = b^2.$$

O20 (*Seniors.*) Find the greatest positive integer n for which the system consisting of n inequalities

$$\begin{cases} 1 < x < 2, \\ 2 < x^2 < 3, \\ \dots\dots\dots \\ n < x^n < n+1 \end{cases}$$

has at least one real solution x .

O21 (*Seniors.*) Let m and n be integers satisfying the following two conditions:

- (1) m and n are either both divisible by 2 or neither is divisible by 2;
- (2) m and n are either both divisible by 3 or neither is divisible by 3.

Prove that the number $mn(m^2 - n^2)(m^4 + n^4 + m^2n^2)$ is divisible by 504.

O22 (*Seniors.*) Real-valued variables satisfy the system of equations

$$\begin{cases} x + y + z + w = 4, \\ x^2 + y^2 + z^2 + w^2 = 6, \\ x^3 + y^3 + z^3 + w^3 = 10, \end{cases}$$

where $x \geq y \geq z \geq w \geq 0$. Find the largest and the least possible value of x .

O23 (*Seniors.*) In a trapezoid $ABCD$, the base AD is longer than the base BC , and $\angle DAB > \angle CDA$. The bisector of the interior angle at vertex A intersects the leg CD at point E , where $\angle EDA > \angle DAE$ and $\angle AED > 90^\circ$. Point F is chosen on the line BE so that $DF \parallel AB$. Point G is chosen on the circumcircle of triangle DEF so that $GE \parallel BC$ ($G \neq E$). Prove that the segments FG and CE have equal length.

Selected Problems from the Final Round of National Olympiad

F1 (Grade 7.) Find the least two natural numbers n for which there exist prime numbers p and q such that both $n + 18$ and $4n + 7$ are divisible by pq .

F2 (Grade 7.) Points A, B, C, D, E, F, G, H are marked on a line in this order. The length of segment AH is 200 cm, the length of segment BG is 85 cm and the length of segment CF is 60 cm. Considering all segments whose endpoints are two marked points, the sum of the lengths of the segments under consideration is 2026 cm. Find the length of segment DE .

F3 (Grade 7.) Let the midpoint of segment DE be A . On the same side of line DE , points B and C are chosen such that triangles ABD and ACE are isosceles with bases AD and AE , respectively, and $\angle CAB = 90^\circ$. On line CA , point F is chosen such that $\angle FDB = 90^\circ$. Knowing that $\angle ABC = 20^\circ$, find the size of angle AFD .

F4 (Grade 7.) There are two types of sticks of the same thickness in the warehouse: blue sticks are 14 cm long and red sticks are x cm long. For transportation, the sticks are arranged lengthwise in rows as short as possible so that each row contains only sticks of one colour and all rows are the same length. Later, all the sticks are cut into pieces as long as possible such that all the pieces are the same length. It is known that from each row 12 pieces of sticks can be made. Find all the possibilities that x can be.

F5 (Grade 8.) Let m and n be integers satisfying

$$2m^2 - 3n^2 + m^2n^2 = 57.$$

Find all possible values of $m + n$.

F6 (Grade 8.) The lengths of sides AD and CD of parallelogram $ABCD$ are correspondingly 4 cm and 9 cm. The bisector of angle BCD intersects lines AB and AD at points L and M . On line BC point E is chosen so that $\angle ELC = 90^\circ$. How many times is the area of parallelogram $ABCD$ bigger than the area of triangle ELM ?

F7 (Grade 8.) A 10×10 board is called a *sea*. A *ship* consists of at least 2 consecutive unit squares of the same row or column of the sea. Unit squares of different ships cannot coincide or have a common side, but a common vertex is allowed. Is it always true that all unit squares of the ships can be colored red or green so that no unit squares having a common side or vertex are of the same color?

F8 (Grade 8.) Let a, b and c be positive integers satisfying the conditions $5a - 2b = 20c$ and $a + 4b = 26c$. How many pairs of positive integers (m, n) exist such that $ma + nb = 2026c$?

F9 (Grade 9.) Find all positive integers n such that the number $3^n - 1$ is the product of two consecutive positive integers.

F10 (Grade 9.) Triangle ABC has a right-angle at vertex A . The point D on the side BC is chosen such that the segments BA and BD are of equal length. Let E be the reflection of the point A across the line BC . The rays BE and AD intersect in point F . Triangle ACF has circumcenter O . Prove that points B, C, F , and O are concyclic.

F11 (Grade 9.) Ats and Pets play the following game on an 8×8 grid of unit squares. At the start all unit squares are uncoloured. On each move a player colours one uncoloured unit square red or blue that has a common side with at most one unit square of that colour. They take turns with Ats starting. The player who cannot make a move loses. Is it possible for either player to beat the opponent in every counterplay, and if so, which one?

F12 (Grade 9.) A point K is chosen on the side AB of the rectangle $ABCD$ and a point L is chosen on the side BC . The segment KL and the diagonal BD of the rectangle intersect in the point M . It is known that the lengths of the segments BM and MD are in the ratio $1 : 4$, while the lengths of the segments KM and ML are in the ratio $2 : 3$. What fraction is the area of triangle BLM of the area of rectangle $ABCD$?

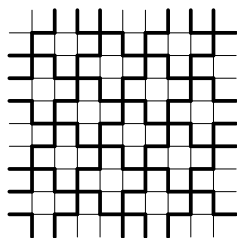
F13 (Grade 10.) Find all pairs (a, b) of positive integers such that both $a^2 - 4b$ and $b^2 - 4a$ are perfect squares.

F14 (Grade 10.) Find all triples (x, y, z) of positive real numbers which satisfy the system of equations

$$\begin{cases} x^{\frac{1}{y}} = z, \\ y^{\frac{1}{z}} = x, \\ z^{\frac{1}{x}} = y. \end{cases}$$

F15 (Grade 10.) In triangle ABC , the circle centered at A and passing through C also passes through the midpoint of BC . The circle centered at B and passing through C , also passes through the midpoint of AC . Prove that ABC is isosceles.

F16 (Grade 10.) The coordinate plane is divided into crosses consisting of 5 unit squares such that each unit square is contained in exactly one cross. Find all pairs (n, m) of positive integers, for which a rectangle with dimensions $n \times m$ can be placed onto the grid such that its sides overlap with the grid lines and no cross has exactly one unit square covered by the rectangle.



F17 (Grade 11.) Let n be the fifth power of a prime. Can the sum of all positive divisors of n be a perfect square?

F18 (Grade 11.) Let n be a positive integer. Order the numbers $(n!)^n$, $n^{n!}$ and $(n^n)!$ by size and find all equality cases.

F19 (Grade 11.) In an acute triangle ABC , the circumcentre is O and the orthocentre is H . The lines AH and BH intersect the circumcircle of ABC at $D \neq A$ and $E \neq B$. Let $K \neq C$ be the intersection of BC and the circumcircle of CEO . Let L be the circumcenter of BHD . Prove that OKL is isosceles.

F20 (Grade 11.) A finite number of positive integers is written on the blackboard. On each turn, two numbers x and y such that $x < y$ are replaced with the numbers $x + \gcd(x+1, y+1) - 1$ and $y - \gcd(x, y)$. Prove that after finitely many turns, all numbers on the blackboard will be equal.

F21 (Grade 12.) Find all triples (x, y, z) of positive integers which satisfy the equation $xy^z = y^{x^z}$.

F22 (Grade 12.) Let a, b, c be real numbers from $[0, 1]$. Prove that

$$9abc + 9 \geq 2(a + b + c)^2.$$

F23 (Grade 12.) A positive integer n is given. Find all positive integers that can be expressed as the sum of one or more distinct positive odd integers, each of which is less than $2n$.

F24 (Grade 12.) Solve the equation

$$\sin 3x + \sin 7x = \sin 10x.$$

Selected Problems from the IMO Team Selection Contests

S1 Let $x_1, x_2, \dots, x_n$ be positive real numbers, where $n \geq 2$. Prove that there exist indices i and j such that

$$x_i < \frac{(x_1 + x_2 + \dots + x_n)^2}{x_1 + 2x_2 + \dots + nx_n} < 2x_j.$$

S2 Let $\mathbb{R}$ be the set of real numbers. For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and for all real numbers x, y and z the following inequality holds:

$$f(x) + f(y) + f(z) \geq f(xy) + f(yz) + f(zx).$$

Prove that f is an even function.

S3 Find all integers $n > 2$ for which there exists a set $\mathcal{P}$ of n points in the plane, such that whenever A and B are distinct points in $\mathcal{P}$, then there exists a point C in $\mathcal{P}$, such that ABC is equilateral.

Problems with Solutions

Selected Problems from Open Contests

O1 (*Juniors.*) Find the number of pairs of positive integers (x, y) for which the equality $20x + 25y = 2025$ holds.

Answer: 20.

Solution: Note that $2025 = 25 \cdot 81$; moving the multiples of 25 to the right hand side in the given equation and factoring, we obtain the equivalent equation $20x = 25(81 - y)$. Dividing both sides of this equation by 5 gives

$$4x = 5(81 - y). \quad (1)$$

It follows that $4x$ must be divisible by 5. Since 5 is a prime number and does not divide 4, the number x must be divisible by 5. Denoting $x = 5a$, we can divide both sides of Eq. (1) by 5, obtaining

$$4a = 81 - y.$$

If $a = 1, 2, \dots, 20$, then $0 < 4a < 81$ and we can find $y = 81 - 4a > 0$. If $a > 20$, then $4a > 81$, so y is negative. Hence there are 20 suitable pairs.

Remark 1: Instead of considering divisibility by 5 in Eq. (1), one may consider divisibility by 4. Namely, Eq. (1) implies that $5(81 - y)$ must be divisible by 4. Since $4 = 2^2$ and $2 \nmid 5$, the number $81 - y$ must be divisible by 4. Thus $y = 4b + 1$, where $b = 0, 1, \dots, 19$, which again gives 20 suitable pairs.

Solution 2: Let $y = 4q + r$, where q and r are integers and $0 \leq r < 4$. Note that a corresponding integer x exists for y if and only if a suitable x exists for r . Indeed, if $20x + 25y = 2025$, then

$$\begin{aligned} 2025 &= 20x + 25y = 20x + 100q + 25y - 100q \\ &= 20(x + 5q) + 25(y - 4q) = 20(x + 5q) + 25r, \end{aligned}$$

and similarly, if $20x + 25r = 2025$, then $2025 = 20(x - 5q) + 25y$. Trying the cases $y = 1$, $y = 2$, $y = 3$ and $y = 4$, we find that a suitable x exists only in the case $y = 1$. Therefore the only positive integers y which can satisfy the given equation are those of the form $4q + 1$. Since the inequality $y \geq 81$ is equivalent to $20x \leq 2025 - 25y = 0$, we have $0 < y < 81$. The answer is therefore the number of integers from 1 to 80 inclusive which leave remainder 1 upon division by 4. There are $\frac{80}{4} = 20$ such numbers.

Remark 2: There is also a solution similar to Solution 2 where one considers the value of x instead of y , and the remainder obtained upon division by 5.

O2 (*Juniors.*) Call a positive integer n *interesting* if it satisfies all of the following conditions:

- (1) the product of the digits of n is 2025;
- (2) the sum of the digits of n is 25;
- (3) no two adjacent digits of the number n are equal;
- (4) the number n is divisible by 5;
- (5) the number n is greater than 2025^2 .

How many interesting positive integers are there?

Answer: 13.

Solution: Since $2025 = 3^4 \cdot 5^2$, the single-digit divisors of the number 2025 are 1, 3, 5 and 9. By condition 1, an interesting number n must consist of these digits. Obviously, the digit 5 must occur exactly twice, and in order to satisfy condition 4 one of these occurrences must be at the end of the number. By condition 2, the sum of the remaining digits must be $25 - 2 \cdot 5 = 15$. Hence the number n cannot contain two digits 9.

- If the number n contains one digit 9, then in order to satisfy condition 1 it must also contain two digits 3. Since $9 + 3 + 3 = 15$, the number n cannot contain any more digits, that is, n is 5-digit. But $2025^2 = 4100625$, so such an n cannot be greater than 2025^2 , which contradicts condition 5.
- If the number n contains no digit 9, then in order to satisfy condition 1 it must contain four digits 3. Since $3 + 3 + 3 + 3 = 12$, it must also contain three digits 1. Such a number is 9-digit, so condition 5 is satisfied. In order to satisfy condition 3, the other digit 5 must be in one of positions 1 to 7 from the left. If it is in the first position, then the remaining digits can only be in the order 3131313. In the remaining 6 cases, the other digits are divided into two parts, one of which has an odd number and the other an even number of digits, and in both parts the digits 3 and 1 must alternate. In the part with an even number of digits, the choice of the first digit is free; in the other part, the first and last digits must be 3. Thus we obtain $1 + 6 \cdot 2 = 13$ possibilities.

It follows that there are 13 interesting numbers in total.

O3 (*Juniors.*) Find all positive integers n which have (not necessarily distinct) positive divisors a and b such that $n = (a + 1)(b + 1)$.

Answer: 4, 6, 12.

Solution: The numbers 4, 6, and 12 are suitable, because:

- the number 4 is divisible by 1 and $4 = 2 \cdot 2 = (1 + 1)(1 + 1)$;
- the number 6 is divisible by 1 and 2 and $6 = 2 \cdot 3 = (1 + 1)(2 + 1)$;
- the number 12 is divisible by 2 and 3 and $12 = 3 \cdot 4 = (2 + 1)(3 + 1)$.

Let us now show that there are no other suitable numbers. Let n be an arbitrary number satisfying the condition of the problem, and let a and b be its divisors such that $n = (a + 1)(b + 1)$. Since $n = (a + 1)(b + 1)$ is divisible by a and the number $a(b + 1)$ is also divisible by a , their difference $(a + 1)(b + 1) - a(b + 1)$, which is $b + 1$, must also be divisible by a . Thus $a \leq b + 1$. Similarly, we can see that the number $a + 1$ is divisible by b , which implies $b \leq a + 1$. For the inequality $a \leq b + 1$, we must have either $a \leq b$ or $a = b + 1$; similarly, for the inequality $b \leq a + 1$, we must have either $b \leq a$ or $b = a + 1$. This yields three cases, which we examine below.

- If $a = b + 1$, then $n = (b + 2)(b + 1) = b^2 + 3b + 2$. The number n , i.e., $b^2 + 3b + 2$, is divisible by b , and the numbers b^2 and $3b$ are also divisible by b . Therefore, the difference $(b^2 + 3b + 2) - b^2 - 3b = 2$ must also be divisible by b . This gives the possibilities $b = 1$ and $b = 2$, from which $n = 6$ or $n = 12$, respectively.
- The case $b = a + 1$ is symmetric and yields the same solutions.
- If $a \neq b + 1$ and $b \neq a + 1$, then both conditions $a \leq b$ and $b \leq a$ must hold, i.e., $a = b$. In this case $n = (b + 1)^2 = b^2 + 2b + 1$. The number n , i.e., $b^2 + 2b + 1$, is divisible by b , and the numbers b^2 and $2b$ are also divisible by b . Therefore, the difference $(b^2 + 2b + 1) - b^2 - 2b = 1$ is also divisible by b . This can happen only if $b = 1$, which yields $n = 4$.

O4 (*Juniors.*) Initially, the number 1 is written on the blackboard. In each step, the number x currently on the blackboard is replaced by the number $\frac{x}{2nx+x+1}$, where n is the step number. Juku makes 44 such steps in a row. Find the number that remains on the blackboard at the end.

Answer: $\frac{1}{45^2}$ or $\frac{1}{2025}$.

Solution: Let us examine the reciprocals of the numbers written on the blackboard. If before step n the reciprocal of some number y , i.e., $\frac{1}{y}$, is on the blackboard, then after step n the number on the blackboard is $\frac{\frac{1}{y}}{\frac{2n}{y} + \frac{1}{y} + 1}$, which is $\frac{1}{2n+1+y}$, the reciprocal of $y + 2n + 1$. Thus, in steps $1, 2, 3, \dots$, the odd numbers $3, 5, 7, \dots$ are added to the denominator. Since the initial number on the blackboard is 1, i.e., $\frac{1}{1}$, after 44 steps the number on the blackboard is $\frac{1}{s}$, where

$$s = 1 + 3 + 5 + \dots + 89.$$

We pair the first term with the last, the second with the second-to-last, and so on, up to the last and first term. The sum of the numbers in each pair is 90, and there are 45 pairs. Since each term appears twice in these pairs, $s = \frac{90 \cdot 45}{2} = 45 \cdot 45 = 2025$. Therefore, the number $\frac{1}{2025}$ remains on the blackboard at the end.

Remark: A direct solution by induction is also possible.

O5 (*Juniors.*) In an acute triangle ABC , the side AB is shorter than the side AC , and the internal angle bisector at vertex A intersects the altitudes BE and CF at points X and Y , respectively. Let Z be the midpoint of the segment XY . Prove that the points A, E, F , and Z lie on the same circle.

Solution: Let H be the orthocenter of triangle ABC , and let K be the intersection point of the internal angle bisector at vertex A with the side BC (Fig. 1). Denote $\angle CAK = \angle KAB = \alpha$. Then

$$\angle YXH = \angle AXE = 180^\circ - \angle XEA - \angle EAX = 180^\circ - 90^\circ - \alpha = 90^\circ - \alpha.$$

Similarly,

$$\angle HYX = \angle FYA = 180^\circ - \angle AFY - \angle YAF = 180^\circ - 90^\circ - \alpha = 90^\circ - \alpha.$$

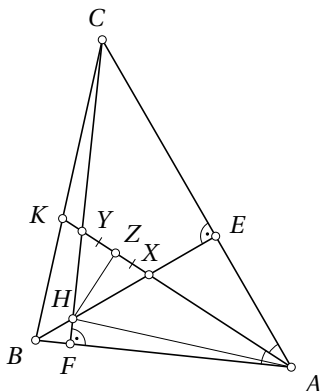


Fig. 1

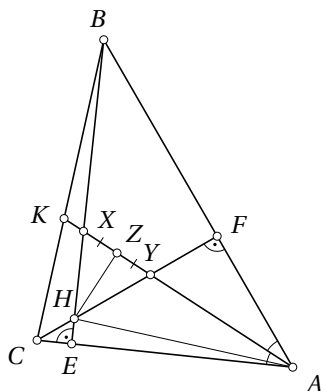


Fig. 2

Thus $\angle YXH = \angle HYZ$, which implies that the triangle HXY is isosceles with the apex at vertex H . This shows that the midpoint Z of the side XY is also the foot of the altitude drawn from vertex H , i.e., $\angle HZA = \angle HZX = 90^\circ$. Therefore, the point Z lies on the circle with diameter AH . However, $\angle AFH = \angle HEA = 90^\circ$ as well, which means that the points E and F also lie on the circle with diameter AH . In conclusion, the points A, E, F , and Z lie on the same circle.

Remark: The statement of the problem also holds if the side AB is longer than the side AC . The points X and Y are located on the segment AK in a different order (Fig. 2), but the same proof works.

O6 (*Juniors.*) The squares of a grid are colored black and white in a chessboard pattern (i.e., any two squares sharing a side have opposite colors). Initially, there is a white queen in one corner square and a black queen in the opposite corner. In one move, a queen can move to any other square in the same row, column, or diagonal. If the opponent's queen is on the path of the moving queen, it is removed from the board, and the player who made the move wins the game. White moves first, and the players take turns. However, each player must move their queen horizontally or vertically on their first move, diagonally on their second move, horizontally or vertically again on their third move, and so on, alternating. Can either player win against any counterplay by their opponent, and if so, who, if:

- the dimensions of the grid are 3×3 ?
- the dimensions of the grid are 4×4 , and an extra rule applies stating that a queen can only move to squares of the color of its starting square?

Answer: a) Yes, Black; b) No.

Solution:

(a) Denote the columns of the grid with the letters a, b, c and the rows with the numbers $1, 2, 3$, so that the white queen is initially on square $a1$ and the black queen on square $c3$ (Fig. 3). We will show that Black can

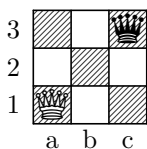


Fig. 3

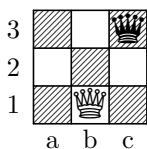


Fig. 4

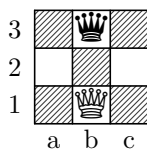


Fig. 5

win regardless of White's moves. On the first move, the white queen can move to square b1 or a2, because by moving to c1 or a3, the opponent could immediately capture the queen on their turn. By symmetry, we may assume that the white queen moves to b1 (Fig. 4). Let the black queen move to b3 (Fig. 5). On the second move, the rules allow White to move to square a2 or c2, but in both cases, Black can capture the queen. Therefore, Black can win the game in any case.

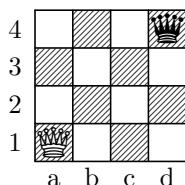


Fig. 6

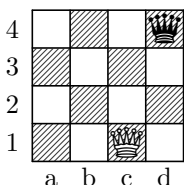


Fig. 7

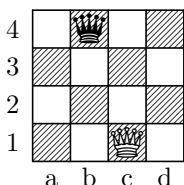


Fig. 8

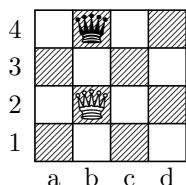


Fig. 9

(b) Denote the columns of the grid with the letters a, b, c, d and the rows with the numbers 1, 2, 3, 4, so that the white queen is initially on square a1 and the black queen on square d4 (Fig. 6). Let us examine how the game can proceed if both players make the best possible moves. On the first move, the white queen can go to square c1 or a3. By symmetry, we may assume that White moves to c1 (Fig. 7). Black can only respond by moving to b4 (Fig. 8), because after the other legal move to d2, White can win on the next move. After the move to b4, White can only move to b2 (Fig. 9), because the other legal moves to a3 and d2 lose immediately. Now Black must choose between the two options a3 and c3, as the move to d2 would lose at once. Suppose Black's second move is to a3 (Fig. 10). We will show that in this case White can win regardless of Black's subsequent actions. Let White's

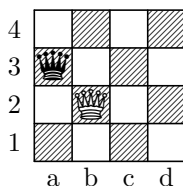


Fig. 10

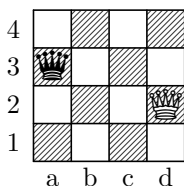


Fig. 11

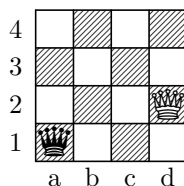


Fig. 12

third move be to d2 (Fig. 11). Black's only response that does not lose immediately is to a1 (Fig. 12). Let White's fourth move be to b4 (Fig. 13). Black's only move that does not lose immediately is to c3 (Fig. 14). White can now move to b2 (Fig. 15). Black's both legal responses to a3 and c1 lose.

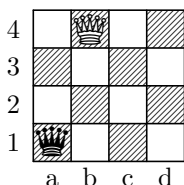


Fig. 13

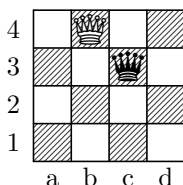


Fig. 14

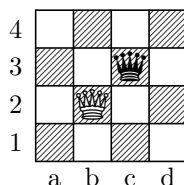


Fig. 15

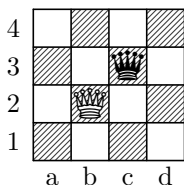


Fig. 16

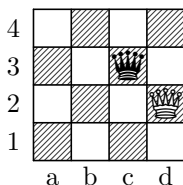


Fig. 17

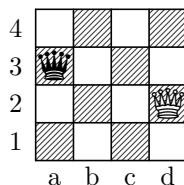


Fig. 18

Therefore, we only need to consider the case where Black's second move is to c3 (Fig. 16). On the third move, White can move to square d2 or b4. By symmetry, we may assume that White moves to d2 (Fig. 17). Black's only response that does not lose immediately is to a3 (Fig. 18). Now the position on the board is essentially the same as after Black's first move, only reflected across the diagonal. Since both players have played optimally, the repetition of the position shows that neither player can win.

Remark: In the variation of part b) where Black's second move is to a3, White can also win by moving to c1 on the fourth move. Black's responses that do not lose immediately are to b2 and d4, but in both cases White can move to c3 and put Black in a fork like in the variation described in the solution.

O7 (*Juniors.*) Find the smallest positive integer n for which the number $2025n$ is the cube of some even number.

Answer: 360.

Solution 1: In the prime factorization of a cube, the exponent of every prime is divisible by 3. In the case of an even number, one of these primes must be 2. Since $2025 = 3^4 \cdot 5^2$, the exponent of the prime 3 must be at least 6, and the exponent of the prime 5 must be at least 3 in the prime factorization of $2025n$. To achieve this, the exponent of the prime 3 must be increased by 2, and the exponent of the prime 5 must be increased by 1. Therefore, the smallest suitable positive integer n is $2^3 \cdot 3^2 \cdot 5$, i.e., 360.

Solution 2: Let $2025n = x^3$, where x is some even number, i.e., x is divisible

by 2. Since 2025 is divisible by 3 and by 5, x^3 is also divisible by 3 and by 5. As both 3 and 5 are prime numbers, it follows that x is divisible by 3 and by 5. Since 2, 3, and 5 are distinct primes, x is also divisible by $2 \cdot 3 \cdot 5 = 30$. Checking the multiples of 30, we see that the solutions to the equations $2025n = 30^3 = 27000$ and $2025n = 60^3 = 216000$ are $n = 13 + \frac{1}{3}$ and $n = 106 + \frac{2}{3}$, respectively, which are not integers. However, the solution to the equation $2025n = 90^3 = 729000$ is $n = 360$, which is an integer. Thus, the smallest positive integer with the required property is $n = 360$.

O8 (*Juniors.*) The cats of Fidgetville all start meowing at the same time. Once they start meowing, they always do so for 1 minute continuously and then rest for 9 minutes, and after 9 minutes of rest, they always start meowing again. The dogs of Fidgetville bark every time someone runs past them, and they also do so for 1 minute, running along with the passerby and stopping after the minute has passed to wait for the next runner.

Juku and Miku run the same circular track, each at a constant speed, starting from the same point. Juku runs clockwise, completing a lap in 30 minutes, while Miku runs counterclockwise, completing a lap in 60 minutes. Initially, all the dogs run with Juku for 1 minute. Juku and Miku stop running when, at the time of a meeting, the dogs are again at the exact same place as them. It is known that at this exact moment, the cats' minute of meowing ends. For how many minutes during Juku and Miku's run did the dogs bark at the same time as the cats were meowing?

Answer: 1.5.

Solution: Juku runs 2 laps per hour, while Miku runs 1 lap in the opposite direction. Therefore, their speed relative to each other is 3 laps per hour. Each meeting takes place after $\frac{1}{3}$ of an hour has passed, at a distance of $\frac{1}{3} = \frac{20}{60}$ of a lap counterclockwise from the previous meeting point. The run does not end until the dogs reach one of the 3 possible meeting points.

Running with Juku, the dogs cover $\frac{1}{30}$ of a lap clockwise, while with Miku they cover $\frac{1}{60}$ of a lap counterclockwise. Thus, as long as the run lasts, the dogs' position shifts by $2 \cdot \frac{1}{30} - \frac{1}{60} = \frac{3}{60}$ of a lap clockwise every hour, although the dogs may have been $\frac{1}{60}$ of a lap further clockwise during the hour. The following table shows the dogs' position at the end of each hour and their furthest position, measured clockwise from the starting point and using one sixtieth of a lap as the unit:

Hour	Position at end of hour	Furthest position	Hour	Position at end of hour	Furthest position
1	3	4	4	12	13
2	6	7	5	15	16
3	9	10	6	18	19

Therefore, the run does not end in the first 6 hours. In the seventh hour, the dogs first move $\frac{1}{30}$ of a lap, i.e., 2 units clockwise with Juku, and reach

exactly 20 units, i.e., $\frac{1}{3}$ of a lap from the starting point. Both Juku and Miku next reach this point after $6 + \frac{2}{3}$ hours, i.e., 400 minutes after the start, at which point the run ends. Since the cats' minute of meowing ends at the exact same moment, the cats always start meowing $10n - 1$ minutes after the run starts and finish meowing $10n$ minutes after the run starts. The following table shows the minutes during which the dogs bark:

Hour	With Juku I	With Juku II	With Miku
1	0–1	31–32	56–57
2	1.5–2.5	32.5–33.5	53–54
3	3–4	34–35	50–51
4	4.5–5.5	35.5–36.5	47–48
5	6–7	37–38	44–45
6	7.5–8.5	38.5–39.5	41–42
7	9–10		

Thus, the cats meow and the dogs bark simultaneously only during the time interval 39–39.5 minutes of the sixth hour and the time interval 9–10 minutes of the seventh hour, totaling 1.5 minutes.

O9 (*Juniors.*) Find all pairs of positive integers (n, m) for which the number $n! + m!$ is not divisible by any prime number greater than 3.

Answer: $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 2)$, $(3, 3)$, $(4, 4)$, $(4, 5)$, $(5, 4)$.

Solution: The condition of the problem means that the equation

$$n! + m! = 2^\alpha 3^\beta \quad (2)$$

holds for some non-negative integers α and β . Without loss of generality, we assume that $n \geq m$. Obviously $m \leq 4$, because if $n \geq m \geq 5$, then 5 would be a divisor of the number $n! + m!$.

Equation (2) is equivalent to the equation

$$m! \left(1 + \frac{n!}{m!} \right) = 2^\alpha 3^\beta.$$

If $n \geq m + 3$, then $\frac{n!}{m!}$ is divisible by $n(n-1)(n-2)$. Among any three consecutive integers, there is both a number divisible by 2 and a number divisible by 3. Thus $\frac{n!}{m!}$ would be divisible by both 2 and 3. But then the number $1 + \frac{n!}{m!}$ would not be divisible by either 2 or 3, which means that this number would have to have some prime factor greater than 3. This contradiction to the condition of the problem shows that $n \leq m + 2$.

It remains to check the pairs (n, m) where $m \leq 4$ and $m \leq n \leq m + 2$. Out of these pairs, $(3, 1)$, $(4, 2)$, $(4, 3)$, $(5, 3)$, and $(6, 4)$ do not satisfy the conditions of the problem, while the remaining ones, i.e., $(1, 1)$, $(2, 1)$, $(2, 2)$, $(3, 2)$, $(3, 3)$, $(4, 4)$, and $(5, 4)$, do. In addition, the pairs with their elements swapped are also suitable.

O10 (*Juniors.*) In the correct addition $A + BC + DEF = GHIJ$, different letters represent different digits. Find all possibilities for the value of $I + J$.

Answer: 8.

Solution: When adding 3 numbers, the carry from any place value to the next can be at most 2. However, since hundreds appear in only one number, the carry into the thousands place can only be 1. Therefore, $G = 1$.

Tens appear in two numbers, and the largest conceivable digits for the tens in them are 9 and 8. Together with a possible carry from the ones, the sum could be at most $9 + 8 + 2 = 19$. Thus, the carry from the tens to the hundreds can also only be 1, and this must be the case in order to trigger a carry into the thousands in turn. This gives $D = 9$ and $H = 0$ as the only option.

Considering the above, the smallest conceivable value for the sum of the ones of the three addends, $A + C + F$, is $2 + 3 + 4 = 9$, and the largest conceivable value is $6 + 7 + 8 = 21$. However, 9, 20, and 21 are not possible, because the digit of the ones of the sum would repeat. Thus a 1 carries over from the ones to the tens. Consequently, $A + C + F = 10 + J$ and $1 + B + E = 10 + I$. Adding these equations yields $A + B + C + E + F = 19 + I + J$. On the other hand,

$$\begin{aligned} A + B + C + E + F &= (0+1+2+3+4+5+6+7+8+9) - D - G - H - I - J \\ &= 45 - 9 - 1 - 0 - I - J = 35 - (I + J). \end{aligned}$$

Solving the system of equations

$$\begin{cases} A + B + C + E + F = 19 + (I + J), \\ A + B + C + E + F = 35 - (I + J) \end{cases}$$

with respect to the unknowns $A + B + C + E + F$ and $I + J$, we get $A + B + C + E + F = 27$ and $I + J = 8$.

The sum $I + J$ can indeed be 8, because by taking, for example, $A = 2$, $B = 4$, $C = 6$, $E = 8$, $F = 7$, $I = 3$, and $J = 5$, we obtain the correct equality $2 + 46 + 987 = 1035$.

Remark: There are 60 possibilities in total, and finding them is not difficult. Since $I + J = 8$, the only candidates for $GHIJ$ are 1026, 1035, 1053, and 1062. If the sum is 1026, then A , C , and F must be either 3, 5, and 8, or 4, 5, and 7 in some order. Correspondingly, B and E must be 4 and 7, or 3 and 8 in some order. In both cases we get 12 possibilities. If the sum is 1035, then A , C , and F can only be 2, 6, and 7 in some order, and B and E can only be 4 and 8 in some order, giving a total of 12 possibilities. If the sum is 1053, then A , C , and F can only be 2, 4, and 7 in some order, and B and E can only be 6 and 8 in some order, giving a total of 12 possibilities. If the sum is 1062, then A , C , and F can only be 3, 4, and 5 in some order, and B and E can only be 7 and 8 in some order. From this we get the final 12 possibilities.

O11 (*Juniors.*) Let AD , BE , and CF be the altitudes of an acute triangle ABC . Point P is the reflection of point F across the line BC . Point Q is the reflection of point P across the line BE .

- (a) Prove that the points P , D , and E lie on the same line.
- (b) Prove that the points Q , F , and E lie on the same line.

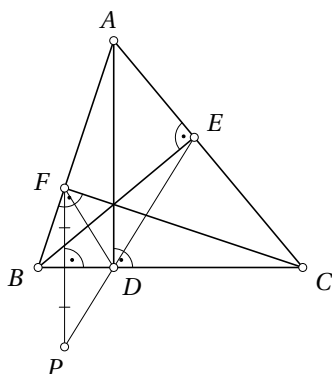


Fig. 19

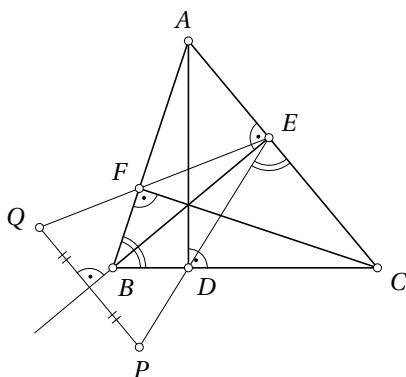


Fig. 20

Solution: Let us denote $\angle CAB = \alpha$ and $\angle ABC = \beta$.

(a) Since $\angle BDA = \angle BEA = 90^\circ$ (Fig. 19), the points D and E lie on the circle with diameter AB . Thus $\angle EDC = \angle CAB = \alpha$. Similarly, the points D and F lie on the circle with diameter AC , and $\angle BDF = \angle CAB = \alpha$. From the definition of P , we also have $\angle PDB = \angle BDF = \alpha$. In conclusion, we obtain $\angle PDB = \angle EDC$, implying that the points P , D , and E are collinear.

(b) Similarly to the beginning of the proof for part a), it can be shown that $\angle FEA = \angle CED = \beta$, and by part a), we then also have $\angle CEP = \beta$ (Fig. 20). By the definition of Q , we have $\angle QEA = \angle CEP = \beta$. In conclusion, $\angle QEA = \angle FEA$, implying that the points Q , F , and E are collinear.

O12 (*Juniors.*) On a 5×5 grid one has to place k identical white tokens and k identical black tokens so that every unit square contains at most one token and when the grid together with the tokens is rotated by 90° around the center of the grid, the arrangement of tokens remains exactly the same.

(a) Find all positive integers k for which this is possible.

(b) How many suitable arrangements are there for the smallest such k ?

Answer: (a) 4, 8, 12; (b) 30.

Solution: We divide the grid into four pieces of sizes 2×3 and 3×2 , so that the central square is left over (Fig. 21). When rotating the grid by 90° around its center, every piece moves to the place where its neighboring piece was initially. Hence whenever there is a token on some square different from

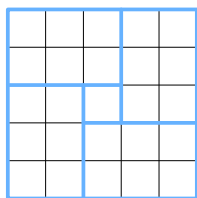


Fig. 21

the central one, there must also be a token of the same color on the corresponding square of the neighboring piece. But then there must also be a token of the same color on the corresponding square of the third and fourth pieces. Therefore, on the squares different from the central one, the number of tokens of each color is divisible by 4. Since the total number of tokens is $2k$, which is even, there cannot be a token on the central square. Hence k is divisible by 4, and on each of the considered pieces there are $\frac{k}{4}$ white and $\frac{k}{4}$ black tokens. Conversely, after placing equal numbers of white and black tokens on one piece, the tokens on the remaining pieces are determined.

(a) Since each part consisting of 6 squares contains $2 \cdot \frac{k}{4} = \frac{k}{2}$ tokens, we must have $\frac{k}{2} \leq 6$. This gives the possibilities $k = 4$, $k = 8$ and $k = 12$.

(b) In the case $k = 4$, on each piece we have to place 1 white and 1 black token. There are 6 ways to place the white token on one piece and then 5 ways to place the black token, for a total of $6 \cdot 5 = 30$ ways. This determines the arrangement of tokens on the whole board, so there are also 30 ways to place 4 white and 4 black tokens according to the requirements.

O13 (*Seniors.*) For each positive integer n , denote by $\tau(n)$ the number of all its positive divisors, and let $\sigma(n)$ denote the sum of all its positive divisors. Find all positive integers n for which $n + \tau(n) = \sigma(n)$.

Answer: 4.

Solution 1: We rewrite the equation in the equivalent form $\tau(n) = \sigma(n) - n$. The right hand side is the sum of all divisors of n except for n . This sum has $\tau(n) - 1$ terms, each of which is at least 1. The left hand side can be rewritten as a sum of ones, where the number of terms is $\tau(n)$. Thus, the equation is satisfied if exactly one term on the right hand side is different from 1, and that term is equal to 2. In other words, the equation is satisfied if and only if the number n has exactly one divisor d for which $1 < d < n$, and $d = 2$. Therefore, n must be an even number greater than 2, and since $\frac{n}{2}$ is also a divisor of n and $1 < \frac{n}{2} < n$, we must have $\frac{n}{2} = 2$. From this, we obtain $n = 4$ as the only solution.

Solution 2: $n = 1$ does not satisfy the equation, since $1 + \tau(1) = 1 + 1 = 2$ and $\sigma(1) = 1$. If n is prime, then $n + \tau(n) = n + 2$, but since $\sigma(n) = n + 1$, there are no prime solutions either. Let n be a composite number and p its smallest prime factor. Then n , $\frac{n}{p}$, and 1 are distinct divisors of n . Therefore $\sigma(n) \geq n + \frac{n}{p} + 1$. On the other hand, $\tau(n) \leq \frac{n}{p} - p + 3$, because the divisors of n can only be the numbers 1 and n , and integers in the interval $\left[p, \frac{n}{p}\right]$. Since $p \geq 2$, we have $n + \tau(n) \leq n + \frac{n}{p} - p + 3 \leq n + \frac{n}{p} + 1 \leq \sigma(n)$. Thus, the number n is a solution to the equation only if equality holds in every inequality used. This means that $p = 2$ and $n, \frac{n}{p}, 1$ must be the only divisors of n . Thus $n = 2^2 = 4$ is the only possibility. Checking shows that $4 + \tau(4) = 4 + 3 = 7 = \sigma(4)$, so $n = 4$ is indeed a solution.

Solution 3: As in Solution 2, we show that neither the number 1 nor any prime number n satisfies the equation. Thus n must have at least 3 divisors. Suppose the number n has exactly 3 divisors. If n were divisible by at least two distinct primes, those primes, 1, and n would all be distinct divisors of n , meaning that n would have more than 3 divisors. However, if n were the cube or a higher power of some prime, then the divisors of n would include more than 3 powers of that prime. Thus $n = p^2$ for some prime number p . The divisors of n are then 1, p , and p^2 , so the condition of the problem gives $p^2 + 3 = 1 + p + p^2$. Simplifying this yields $p = 2$, which means $n = 2^2 = 4$. Checking shows that $4 + \tau(4) = 4 + 3 = 7 = \sigma(4)$, so $n = 4$ indeed satisfies the equation.

Consider now an arbitrary number n satisfying the condition. All divisors can be grouped into pairs $(d, \frac{n}{d})$, whose members can be equal only if $d = \sqrt{n}$. The smaller of the distinct numbers d and $\frac{n}{d}$ must also be smaller than $\sqrt{n}$, otherwise the product of these divisors would be greater than n , a contradiction. Thus, there can be at most $\sqrt{n}$ such pairs of divisors, meaning the number of divisors is less than $2\sqrt{n}$. Consequently, $n + \tau(n) < n + 2\sqrt{n}$. On the other hand, if the number n had at least 4 divisors, there would have to be a divisor d such that $d \neq 1$, $d \neq n$, and $d \neq \sqrt{n}$. Then $d \neq \frac{n}{d}$ as well, which yields

$$d + \frac{n}{d} - 2\sqrt{n} = (\sqrt{d})^2 + \left(\frac{\sqrt{n}}{\sqrt{d}}\right)^2 - 2\sqrt{d} \cdot \frac{\sqrt{n}}{\sqrt{d}} = \left(\sqrt{d} - \frac{\sqrt{n}}{\sqrt{d}}\right)^2 > 0,$$

from which $d + \frac{n}{d} > 2\sqrt{n}$ and in turn $\sigma(n) > n + d + \frac{n}{d} > n + 2\sqrt{n}$. In conclusion, $n + \tau(n) < \sigma(n)$, which means the condition of the problem could not hold. Therefore, there are no other solutions.

O14 (*Seniors.*) Solve the following system in real numbers:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 + 7x_7 = 1, \\ x_2 + 2x_3 + 3x_4 + 4x_5 + 5x_6 + 6x_7 + 7x_1 = 2, \\ x_3 + 2x_4 + 3x_5 + 4x_6 + 5x_7 + 6x_1 + 7x_2 = 3, \\ x_4 + 2x_5 + 3x_6 + 4x_7 + 5x_1 + 6x_2 + 7x_3 = 4, \\ x_5 + 2x_6 + 3x_7 + 4x_1 + 5x_2 + 6x_3 + 7x_4 = 5, \\ x_6 + 2x_7 + 3x_1 + 4x_2 + 5x_3 + 6x_4 + 7x_5 = 6, \\ x_7 + 2x_1 + 3x_2 + 4x_3 + 5x_4 + 6x_5 + 7x_6 = 7, \end{cases}$$

where the variables are $x_1, x_2, x_3, x_4, x_5, x_6, x_7$.

Answer: $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = \frac{2}{7}, x_7 = -\frac{5}{7}$.

Solution: Adding all the equations together followed by dividing both sides by $1 + 2 + 3 + 4 + 5 + 6 + 7$ gives

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 1. \quad (3)$$

Subtracting equation k from equation $k + 1$, we obtain

$$6x_k - x_{k+1} - \dots - x_7 - x_1 - \dots - x_{k-1} = 1. \quad (4)$$

This holds for each $k = 1, 2, 3, 4, 5, 6$. Adding Equations (3) and (4), we get $7x_k = 2$, so $x_k = \frac{2}{7}$ for each $k = 1, 2, 3, 4, 5, 6$. Substituting these values into Eq. (3), we get $6 \cdot \frac{2}{7} + x_7 = 1$, from which $x_7 = 1 - 6 \cdot \frac{2}{7} = -\frac{5}{7}$.

O15 (*Seniors.*) Can the number 2025 be expressed as the difference of the cubes of two integers?

Answer: No.

Solution 1: Suppose that for some integers a and b

$$a^3 - b^3 = 2025. \quad (5)$$

Then $a^3 > b^3$, from which it follows that $a > b$. Let $a = b + x$, where $x > 0$. Since $a^3 - b^3 = (b + x)^3 - b^3 = 3b^2x + 3bx^2 + x^3$, Eq. (5) takes the form

$$3b^2x + 3bx^2 + x^3 = 2025.$$

Since $3b^2x$, $3bx^2$, and 2025 are divisible by 3, x^3 and consequently x must also be divisible by 3. Substituting $x = 3y$ and dividing by 9, we get

$$b^2y + 3by^2 + 3y^3 = 225. \quad (6)$$

Since $3by^2$, $3y^3$, and 225 are divisible by 3, b^2y must also be divisible by 3. Consequently, either b or y must be divisible by 3. If b were divisible by 3, then b^2y , $3by^2$, and 225 would all be divisible by 9, meaning that $3y^3$ would also have to be divisible by 9. It would follow that y^3 and consequently y must be divisible by 3. But then b^2y , $3by^2$, and $3y^3$ would all be divisible by 27, meaning that 225 would also have to be divisible by 27, a contradiction. Thus b is not divisible by 3, and y must be divisible by 3. Now, since $3by^2$, $3y^3$, and 225 are divisible by 9, b^2y must also be divisible by 9. Since b is not divisible by 3, y must be divisible by 9. Equation (6) can be equivalently rewritten as a quadratic equation with respect to b in the form

$$y b^2 + 3y^2 b + (3y^3 - 225) = 0.$$

The discriminant of this quadratic equation is $-3y^4 + 900y = 3y(300 - y^3)$. Since $y > 0$ and y is divisible by 9, we have $y^3 \geq 9^3 > 300$, so the discriminant is always negative. Thus, the equation has no solutions.

Solution 2: Suppose that for some integers a , b Eq. (5) holds. Then $a^3 > b^3$, from which it follows that $a > b$.

If a and b were of opposite signs, then we would have $a > 0 > b$, and Eq. (5) could be equivalently rewritten in the form $a^3 + |b|^3 = 2025$. Since both terms are positive here, both are less than the sum 2025, while one of them must account for at least half of the sum. Half of the sum 2025 is greater than 1000, i.e., 10^3 . Calculating, we find $11^3 = 1331$, $12^3 = 1728$, and $13^3 = 2197 > 2025$, so only 1331 and 1728 remain as possible cube values. Since $2025 - 1331 = 694$ and $2025 - 1728 = 297$, and neither 694 nor 297 is a perfect cube, the given equation has no solutions with opposite signs.

We now show that there are no other solutions either. Without loss of generality, we assume that $a > b \geq 0$. Let $a = b + x$, where $x > 0$. From

Eq. (5) we get $2025 = (b+x)^3 - b^3 = x(3b^2 + 3bx + x^2)$. Therefore, x must be a divisor of the number 2025. The smallest positive divisors of 2025 are 1, 3, 5, 9, 15. If x were not divisible by 3, then x^2 would not be divisible by 3, but then $3b^2 + 3bx + x^2$ would not be divisible by 3 either. Thus, the product $x(3b^2 + 3bx + x^2)$ would lack the prime factor 3. This would be a contradiction, as 2025 is divisible by 3. Consequently, x is divisible by 3.

- If $x = 3$, then the equation $2025 = x(3b^2 + 3bx + x^2)$ reduces to $225 = b^2 + 3b + 3$, which in turn rearranges into $b^2 + 3b - 222 = 0$. The discriminant of this quadratic equation is 897. Since it is not the square of an integer, the equation has no integer solutions.
- If $x = 9$, then the equation $2025 = x(3b^2 + 3bx + x^2)$ reduces to $75 = b^2 + 9b + 27$, which in turn rearranges into $b^2 + 9b - 48 = 0$. The discriminant of this quadratic equation is 273. Since it is not the square of an integer, the equation has no integer solutions.
- If $x \geq 15$, then $x(3b^2 + 3bx + x^2) \geq x \cdot x^2 = x^3 \geq 15^3 > 2025$, a contradiction.

Consequently, the given equation has no integer solutions.

Solution 3: Suppose that for some integers a, b Eq. (5) holds. Then $a^3 > b^3$, from which it follows that $a > b$. As shown in Solution 2, there are no solutions with opposite signs.

We now show that there are no other solutions either. Without loss of generality, we assume that $a > b \geq 0$. Since 2025 is divisible by 3 and 5, Eq. (5) yields $a^3 \equiv b^3 \pmod{3}$ and $a^3 \equiv b^3 \pmod{5}$. Note that $0^3 \equiv 0 \pmod{3}$, $1^3 \equiv 1 \pmod{3}$, and $2^3 \equiv 2 \pmod{3}$, so the relation $a^3 \equiv b^3 \pmod{3}$ also implies the relation $a \equiv b \pmod{3}$. Furthermore, note that $0^3 \equiv 0 \pmod{5}$, $1^3 \equiv 1 \pmod{5}$, $2^3 \equiv 3 \pmod{5}$, $3^3 \equiv 2 \pmod{5}$, and $4^3 \equiv 4 \pmod{5}$; thus if we had $a \not\equiv b \pmod{5}$, we would also have $a^3 \not\equiv b^3 \pmod{5}$, a contradiction. Consequently, $a \equiv b \pmod{5}$. In conclusion, we obtain $a \equiv b \pmod{15}$. Since $a \neq b$, we have $a - b \geq 15$, and moreover $a \geq 15$. Consequently, $a^2 + ab + b^2 \geq 15^2$, which gives

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \geq 15^3 > 2025.$$

Therefore, Eq. (5) cannot be satisfied.

O16 (*Seniors.*) Does there exist a function defined on all real numbers and taking only real values, which satisfies the equation

$$f(x) + f(f(y)) + f(f(f(x+y))) = x + y$$

for all real numbers x and y ?

Answer: No.

Solution 1: Denote by $P(x, y)$ the equation of the problem. Then $P(x, f(x))$ yields

$$f(x) + f(f(f(x))) + f(f(f(x + f(x)))) = x + f(x), \quad (7)$$

while $P(f(x), x)$ yields

$$f(f(x)) + f(f(x)) + f(f(f(f(x) + x))) = f(x) + x. \quad (8)$$

Combining Equations (7) and (8), we get

$$f(x) + f(f(f(x))) = 2f(f(x)). \quad (9)$$

On the other hand, $P(x, 0)$ gives us $f(x) + f(f(0)) + f(f(f(x))) = x$, i.e.,

$$f(x) + f(f(f(x))) = x - f(f(0)). \quad (10)$$

Combining Equations (9) and (10), we get

$$2f(f(x)) = x - f(f(0)). \quad (11)$$

Substituting $x = 0$ into Eq. (11) followed by combining like terms yields $3f(f(0)) = 0$, from which $f(f(0)) = 0$. Thus, Eq. (11) takes the form $2f(f(x)) = x$, i.e.,

$$f(f(x)) = \frac{1}{2}x. \quad (12)$$

This must hold for every real number x . Denote Eq. (12) by $Q(x)$. Now using $Q(f(x))$ in Eq. (10), we get $f(x) + \frac{1}{2}f(x) = x$, which simplifies to $f(x) = \frac{2}{3}x$. From this identity, however, it follows that

$$f(f(x)) = f\left(\frac{2}{3}x\right) = \frac{2}{3} \cdot \frac{2}{3}x = \frac{4}{9}x. \quad (13)$$

This must also hold for every real number x . But in general $\frac{1}{2}x \neq \frac{4}{9}x$, which means that Equations (12) and (13) contradict each other. Therefore, no function satisfies the condition of the problem.

Solution 2: We equivalently rewrite the given equation in the form

$$f(x) + f(f(y)) = (x + y) - f(f(f(x + y))). \quad (14)$$

Swapping x and y , the right hand side of Eq. (14) remains the same. Therefore, the left hand side also does not change, i.e.,

$$f(x) + f(f(y)) = f(y) + f(f(x)). \quad (15)$$

Substituting $y = 0$ into Eq. (15) and rearranging the terms, we obtain

$$f(f(x)) = f(x) + f(f(0)) - f(0). \quad (16)$$

This must hold for every real number x ; let $R(x)$ denote Eq. (16).

On the other hand, substituting $y = 0$ into the original equation, we get

$$f(x) + f(f(0)) + f(f(f(x))) = x.$$

Applying $R(f(x))$ and then $R(x)$ here, we obtain

$$f(x) + f(f(0)) + f(x) + 2(f(f(0)) - f(0)) = x.$$

From this we can express

$$f(x) = \frac{1}{2}x + f(0) - \frac{3}{2}f(f(0)). \quad (17)$$

This must hold for every real number x ; let $S(x)$ denote Eq. (17). But now the left hand sides of Eq. (16) and $S(f(x))$ are equal; therefore, the right hand sides must also be equal:

$$f(x) + f(f(0)) - f(0) = \frac{1}{2}f(x) + f(0) - \frac{3}{2}f(f(0)),$$

from which we can express $f(x) = 4f(0) - 5f(f(0))$. Consequently, f is a constant function. However, this contradicts Eq. (17). The contradiction shows that no function satisfies the condition of the problem.

O17 (*Seniors.*) In an acute triangle ABC , the side AB is shorter than the side AC , and the incenter is I . Let J be the reflection of point I across the line BC . The segment AJ intersects the circumcircles of triangles BIJ and CIJ at points D and E , respectively, with $D \neq J$ and $E \neq J$. A circle with center K is tangent to the side DE of triangle IDE and to the extensions of its other two sides. Prove that the points B, K, I , and C are concyclic.

Solution: The incenter I of triangle ABC is the intersection point of the internal angle bisectors of this triangle. From the definition of point K , DK and EK are the external angle bisectors of triangle IDE (Fig. 22) and are therefore perpendicular to the corresponding internal angle bisectors. Consequently,

$$\angle KDE = 90^\circ - \frac{1}{2}\angle EDI = 90^\circ - \frac{1}{2}\angle JDI = 90^\circ - \frac{1}{2}\angle JBI = 90^\circ - \angle CBI,$$

$$\angle DEK = 90^\circ - \frac{1}{2}\angle IED = 90^\circ - \frac{1}{2}(180^\circ - \angle JEI) = 90^\circ - \frac{1}{2}\angle ICJ = 90^\circ - \angle ICB.$$

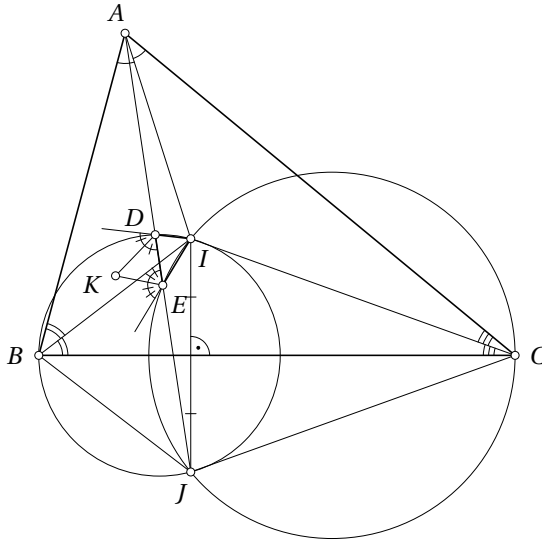


Fig. 22

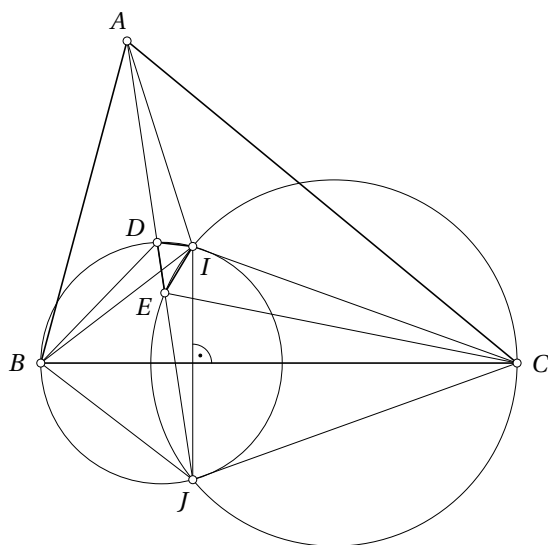


Fig. 23

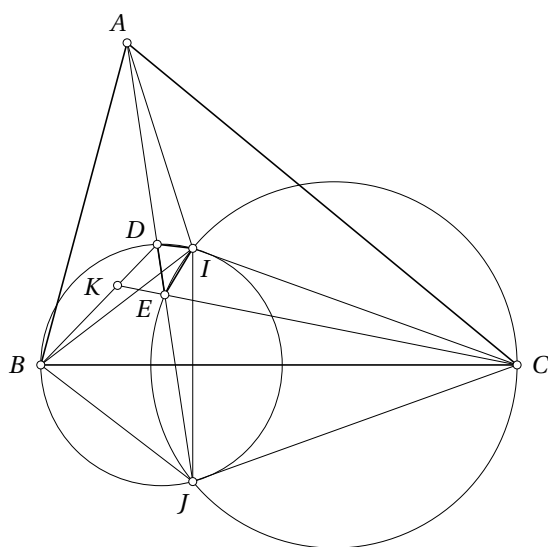


Fig. 24

At the same time,

$$\angle BDE = \angle BDJ = \angle BIJ = 90^\circ - \angle CBI,$$

$$\angle CED = 180^\circ - \angle JEC = 180^\circ - \angle JIC = 180^\circ - (90^\circ - \angle ICB) = 90^\circ + \angle ICB$$

(Fig. 23). In conclusion, $\angle KDE = \angle BDE$ and $\angle DEK + \angle CED = 180^\circ$, which shows that the points B, K , and D are collinear, and the points C, E , and K are also collinear (Fig. 24). It follows that

$$\angle IBK = \angle IBD = \angle IJD = \angle IJE = \angle ICE = \angle ICK,$$

where we use directed angles to cover both the case where point K lies on the segment BD and the case where point K lies on the extension of segment BD beyond point B . The obtained equality shows that the points B, K, I , and C lie on the same circle.

Remark: Due to a jury error, the assumption that triangle ABC is acute was missing from the problem text presented at the competition. This allowed for a situation where $B = D$ (Fig. 25) and in which the above solution does not work, because one cannot speak of the angle BDE .

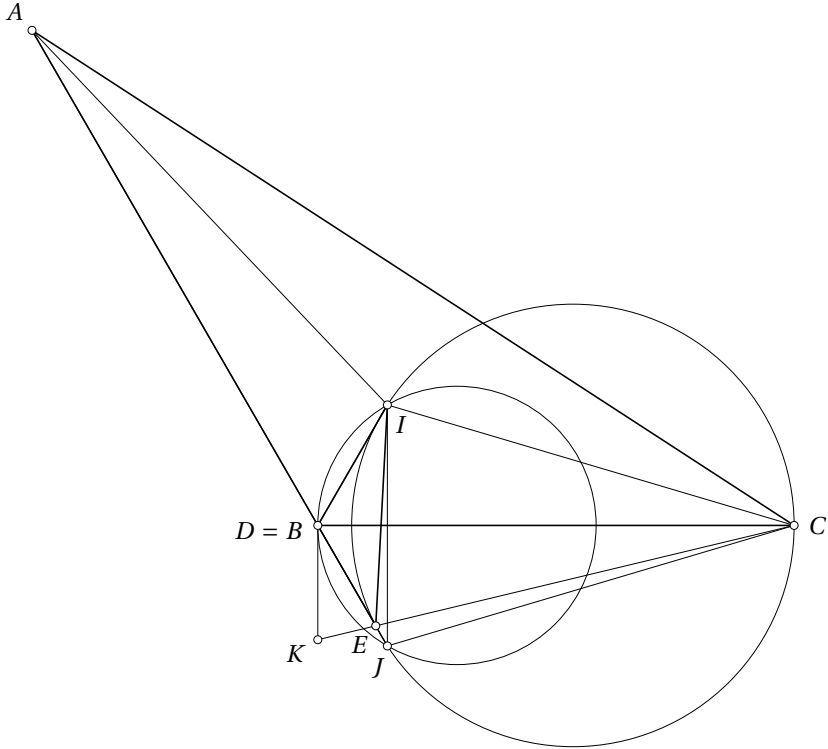


Fig. 25

O18 (*Seniors.*) On a 7×7 grid, the centers of some squares are marked such that there is no rectangle whose vertices are all marked points and whose sides are parallel to the grid lines. Find the greatest possible number of marked points.

Answer: 21.

Solution 1: If it were possible to mark at least 22 points, then, by the pigeon-hole principle, there would be at least 4 marked points in one column. Let us look at the 4 rows containing these marked points. In each of the remaining 6 columns, there could be at most 1 point in these rows. Therefore, in these rows, there could be at most $4 + 6 = 10$ points in total.

Let us now look at the remaining 3 rows. There would have to be a total of at least $22 - 10 = 12$ marked points in them. If there is a column that has a marked point in all of these rows, then none of the remaining 6 columns can have more than 1 point in these rows. In total, at most $3 + 6 = 9$ marked points could lie in these rows. Thus, no column has more than 2 marked points in these rows. Since there are only 3 distinct ways to choose 2 rows out of the 3 available, at most 3 columns can have 2 marked points, and the remaining columns can have at most 1 marked point. Therefore, a total of at most $7 + 3 = 10$ points can be marked.

This shows that it is not possible to mark more than 21 points. Figure 26 shows that it is possible to mark 21 points.

Solution 2: Suppose the centers of some squares are marked in the required manner. For each $i = 1, \dots, 7$, let exactly n_i squares be marked in the i -th column. Then for each $i = 1, \dots, 7$, the number of pairs of marked squares in the i -th column is exactly $C_{n_i}^2 = \frac{n_i(n_i-1)}{2}$. Hence, the total number of pairs of marked squares located in the same column is $\frac{n_1(n_1-1)}{2} + \dots + \frac{n_7(n_7-1)}{2}$. By the condition of the problem, no two counted pairs of squares can lie in the same pair of rows. Thus, the number of counted pairs does not exceed the number of pairs of distinct rows, which is $C_7^2 = \frac{7 \cdot 6}{2} = 21$. Consequently,

$$\frac{n_1(n_1-1)}{2} + \dots + \frac{n_7(n_7-1)}{2} \leq 21.$$

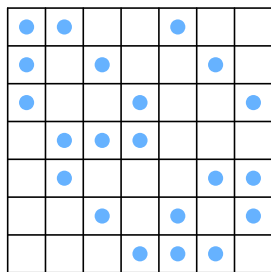


Fig. 26

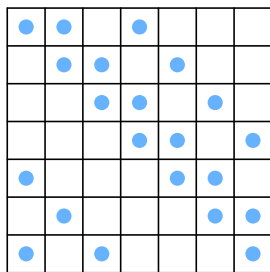


Fig. 27

If the total number of marked points were at least 22, i.e., $n_1 + \dots + n_7 \geq 22$, then using the QM-AM inequality, we would obtain

$$\begin{aligned} \frac{n_1(n_1-1)}{2} + \dots + \frac{n_7(n_7-1)}{2} &= \frac{1}{2} (n_1^2 + \dots + n_7^2) - \frac{1}{2} (n_1 + \dots + n_7) \\ &\geq \frac{1}{14} (n_1 + \dots + n_7)^2 - \frac{1}{2} (n_1 + \dots + n_7) \\ &= \frac{1}{14} (n_1 + \dots + n_7) ((n_1 + \dots + n_7) - 7) \\ &\geq \frac{22 \cdot 15}{14} > 21, \end{aligned}$$

a contradiction.

On the other hand, if we mark the centers of the squares on the first, second, and fourth cyclic diagonals (Fig. 27), 21 points are marked and the conditions of the problem are satisfied. Thus, the greatest possible number of marked points is 21.

O19 (*Seniors.*) Find all pairs of integers (a, b) for which

$$a^3 - a = b^2.$$

Answer: $(-1, 0), (0, 0), (1, 0)$.

Solution 1: Factoring the left hand side, we get the equivalent equation

$$(a^2 - 1)a = b^2. \quad (18)$$

Since the consecutive integers $a^2 - 1$ and a^2 are coprime, the numbers $a^2 - 1$ and a are also coprime. Their product is a perfect square by Eq. (18). Therefore the numbers $|a^2 - 1|$ and $|a|$ must be perfect squares. But a^2 is also a perfect square, while $|a^2 - 1|$ and a^2 are consecutive integers. The only two consecutive integers which are both perfect squares are 0 and 1. Hence $|a^2 - 1|$ and a^2 must be 0 and 1 in one order or the other. If $|a^2 - 1| = 0$ and $a^2 = 1$, then there are two possibilities, $a = 1$ and $a = -1$. If $|a^2 - 1| = 1$ and $a^2 = 0$, then $a = 0$. In every case, Eq. (18) gives $b^2 = 0$, so $b = 0$.

Solution 2: Factoring the left hand side, we get the equivalent equation

$$(a - 1)a(a + 1) = b^2. \quad (19)$$

We consider three cases.

- If $a < -1$, then all factors on the left hand side of Eq. (19) are negative, and therefore the whole left hand side is negative. The right hand side, however, is nonnegative. Thus there are no solutions.
- Substituting $a = -1, 0, 1$ into Eq. (19), we get $b = 0$ each time. This gives the solutions $(-1, 0), (0, 0)$ and $(1, 0)$.
- If $a > 1$, then all factors on the left hand side of Eq. (19) are positive. Since $\gcd(a, a - 1) = 1$ and $\gcd(a, a + 1) = 1$, we must also have $\gcd(a, (a - 1)(a + 1)) = 1$, and since the product of the numbers a and $(a - 1)(a + 1)$ is the square of an integer, both a and $(a - 1)(a + 1)$ must be squares of integers.

Note that $\gcd(a-1, a+1) = \gcd(a-1, 2) \leq 2$. If $\gcd(a-1, a+1) = 1$, then both $a-1$ and $a+1$ must be squares of integers, but three consecutive integers $a-1, a, a+1$ cannot all be squares of integers. If $\gcd(a-1, a+1) = 2$, then $\frac{a-1}{2} \cdot \frac{a+1}{2}$ is also the square of an integer, while $\gcd\left(\frac{a-1}{2}, \frac{a+1}{2}\right) = 1$. Hence $\frac{a-1}{2}$ and $\frac{a+1}{2}$ are both squares of integers. But two consecutive positive integers $\frac{a-1}{2}, \frac{a+1}{2}$ cannot both be squares of integers. No further solutions arise from this case.

O20 (*Seniors.*) Find the greatest positive integer n for which the system consisting of n inequalities

$$\begin{cases} 1 < x < 2, \\ 2 < x^2 < 3, \\ \dots\dots\dots \\ n < x^n < n+1 \end{cases}$$

has at least one real solution x .

Answer: 4.

Solution 1: We can write the given system of inequalities in the equivalent form

$$\begin{cases} 1 < x < 2, \\ \sqrt{2} < x < \sqrt{3}, \\ \dots\dots\dots \\ \sqrt[n]{n} < x < \sqrt[n]{n+1}. \end{cases}$$

All real numbers x satisfying the second inequality also satisfy the first inequality, so the first inequality may be omitted. Since $\sqrt[4]{4} = \sqrt{\sqrt{4}} = \sqrt{2}$ and $\sqrt[4]{5} = \sqrt{\sqrt{5}} < \sqrt{3}$, all real numbers x satisfying the fourth inequality also satisfy the second inequality. Hence the second inequality may also be omitted. Next we notice that $\sqrt[5]{6} < \sqrt[3]{3}$, because $\sqrt[5]{6} = \sqrt[15]{6^3} = \sqrt[15]{216}$ and $\sqrt[3]{3} = \sqrt[15]{3^5} = \sqrt[15]{243}$. Therefore the third and fifth inequalities contradict each other, so for $n \geq 5$ there are no solutions. It remains to show that the third and fourth inequalities have common solutions. For this it is enough to verify that $\sqrt[4]{4} < \sqrt[3]{3} < \sqrt[4]{5} < \sqrt[3]{4}$. Indeed, $\sqrt[4]{4} = \sqrt[12]{4^3} = \sqrt[12]{64}$, $\sqrt[3]{3} = \sqrt[12]{3^4} = \sqrt[12]{81}$, $\sqrt[4]{5} = \sqrt[12]{5^3} = \sqrt[12]{125}$ and $\sqrt[3]{4} = \sqrt[12]{4^4} = \sqrt[12]{256}$, and $64 < 81 < 125 < 256$. Thus in the case $n = 4$, the given system of inequalities has a real solution x .

Solution 2: Suppose that a real number x satisfies the third and fifth inequalities. Then in particular $3 < x^3$ and $x^5 < 6$, which are equivalent to the inequalities $3^5 < x^{15}$ and $x^{15} < 6^3$, respectively. From this we get $243 = 3^5 < 6^3 = 216$, a contradiction. Hence for $n \geq 5$ there are no solutions. It remains to show that the first four inequalities have a common solution. The number $x = \frac{13}{9} = 1 + \frac{4}{9}$ works, because $x^2 = \frac{169}{81} = 2 + \frac{7}{81}$, $x^3 = \frac{2197}{729} = 3 + \frac{10}{729}$ and $x^4 = \frac{28561}{6561} = 4 + \frac{2317}{6561}$.

O21 (Seniors.) Let m and n be integers satisfying the following two conditions:

- (1) m and n are either both divisible by 2 or neither is divisible by 2;
- (2) m and n are either both divisible by 3 or neither is divisible by 3.

Prove that the number $mn(m^2 - n^2)(m^4 + n^4 + m^2n^2)$ is divisible by 504.

Solution 1: Since $504 = 7 \cdot 8 \cdot 9$ and the factors are pairwise coprime, it is enough to show divisibility by 7, 8 and 9.

If m and n are both divisible by 2, then $4 \mid mn$ and the remaining two factors are also even, so the given product is divisible by 8 (in fact, even by 16). If neither m nor n is divisible by 2, then $m^2 \equiv n^2 \equiv 1 \pmod{8}$, hence $8 \mid m^2 - n^2$. It follows that the given product is again divisible by 8.

If m and n are both divisible by 3, then $9 \mid mn$. If neither m nor n is divisible by 3, then $m^2 \equiv n^2 \equiv 1 \pmod{3}$, hence $3 \mid m^2 - n^2$. We also get $m^4 \equiv n^4 \equiv m^2n^2 \equiv 1 \pmod{3}$, which yields $3 \mid m^4 + n^4 + m^2n^2$. Therefore the given product is again divisible by 9.

Finally, we examine divisibility by 7. If m or n is divisible by 7, then the product is divisible by 7. Now assume that neither m nor n is divisible by 7. Then the remainder of m^2 upon division by 7 is 1, 2 or 4, and the same is true for the remainder of n^2 upon division by 7. If $m^2 \equiv n^2 \pmod{7}$, then $7 \mid m^2 - n^2$ and the product is divisible by 7. But if $m^2 \not\equiv n^2 \pmod{7}$, then m^4, n^4 and m^2n^2 have remainders 1, 2, 4 modulo 7 in some order, so $7 \mid m^4 + n^4 + m^2n^2$ and the product is divisible by 7.

Solution 2: Since $504 = 7 \cdot 8 \cdot 9$ and the factors are pairwise coprime, it is enough to show divisibility by 7, 8 and 9. Notice that

$$(m^2 - n^2)(m^4 + n^4 + m^2n^2) = m^6 - n^6,$$

so it is enough to analyze the expression $mn(m^6 - n^6)$.

If m and n are both divisible by 2, then $4 \mid mn$ and $2 \mid m^6 - n^6$, so the whole product is divisible by 8. If neither m nor n is divisible by 2, then $m^6 - n^6$ is divisible by 8 as a difference of squares of odd numbers. Again the product is divisible by 8.

If m and n are both divisible by 3, then $9 \mid mn$ and the whole product is divisible by 9. If neither m nor n is divisible by 3, then by Euler's theorem $m^6 \equiv n^6 \equiv 1 \pmod{9}$, hence $9 \mid m^6 - n^6$.

If m or n is divisible by 7, then $7 \mid mn$ and the whole product is divisible by 7. If neither m nor n is divisible by 7, then by Fermat's little theorem $m^6 \equiv n^6 \equiv 1 \pmod{7}$, hence $7 \mid m^6 - n^6$.

O22 (Seniors.) Real-valued variables satisfy the system of equations

$$\begin{cases} x + y + z + w = 4, \\ x^2 + y^2 + z^2 + w^2 = 6, \\ x^3 + y^3 + z^3 + w^3 = 10, \end{cases}$$

where $x \geq y \geq z \geq w \geq 0$. Find the largest and the least possible value of x .

Answer: greatest 2, least $1 + \frac{\sqrt{2}}{2}$.

Solution 1: Let $x = 1 + a$, $y = 1 + b$, $z = 1 + c$ and $w = 1 + d$. Substituting into the equations of the given system and simplifying, we get

$$\begin{cases} a + b + c + d = 0, \\ a^2 + b^2 + c^2 + d^2 + 2(a + b + c + d) = 2, \\ a^3 + b^3 + c^3 + d^3 + 3(a^2 + b^2 + c^2 + d^2) + 3(a + b + c + d) = 6. \end{cases}$$

Using the first equation of the new system in the second, and then the first and second equations in the third, we can simplify the system to the form

$$\begin{cases} a + b + c + d = 0, \\ a^2 + b^2 + c^2 + d^2 = 2, \\ a^3 + b^3 + c^3 + d^3 = 0. \end{cases} \quad (20)$$

Here from the first equation we get $c + d = -(a + b)$, so the third equation of system (20) can be written in the form

$$(a + b)(a^2 - ab + b^2) - (a + b)(c^2 - cd + d^2) = 0. \quad (21)$$

If $a + b = 0$, then also $c + d = 0$, which in view of the order $a \geq b \geq c \geq d$ would be possible only if $a = b = c = d = 0$, but this does not satisfy the second equation of system (20). Hence we can reduce equation (21) to the form $(a^2 - ab + b^2) - (c^2 - cd + d^2) = 0$. Adding to this the second equation of system (20), we get

$$2a^2 + 2b^2 - ab + cd = 2. \quad (22)$$

On the other hand, squaring both sides of the equality $c + d = -(a + b)$ and using the obtained relation in the second equation of system (20), we get

$$a^2 + b^2 + ab - cd = 1. \quad (23)$$

Adding the corresponding sides of Equations (22) and (23), we can simplify the result to $a^2 + b^2 = 1$. Therefore $a \leq 1$, and since $a \geq b$, we have $a^2 \geq \frac{1}{2}$, i.e., $a \geq \frac{\sqrt{2}}{2}$. This corresponds to $2 \geq x \geq 1 + \frac{\sqrt{2}}{2}$. The extreme values are realized by the solutions $(x, y, z, w) = (2, 1, 1, 0)$ and $(x, y, z, w) = (1 + \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2})$ of the original system of equations.

Solution 2: To find the greatest value, we consider the function

$$f(t) = t(t-1)^2 = t^3 - 2t^2 + t$$

on the interval $t \in [0, \infty)$. Clearly $f(t) \geq 0$, and in the case $t > 2$ we have $f(t) > 2 \cdot (2-1)^2 = 2$, so if $f(t) \leq 2$, then $t \leq 2$. Now we compute

$$\begin{aligned} f(x) + f(y) + f(z) + f(w) &= (x^3 + y^3 + z^3 + w^3) - 2 \cdot (x^2 + y^2 + z^2 + w^2) + (x + y + z + w) \\ &= 10 - 2 \cdot 6 + 4 = 2. \end{aligned}$$

Thus $f(x) \leq 2$, and by the above $x \leq 2$. The value 2 is attainable by taking $(x, y, z, w) = (2, 1, 1, 0)$.

To find the least value, we consider the functions

$$g(t) = \left(t - \left(1 - \frac{\sqrt{2}}{2}\right)\right) \left(t - \left(1 + \frac{\sqrt{2}}{2}\right)\right) = (t-1)^2 - \frac{1}{2} = t^2 - 2t + \frac{1}{2},$$

$$h(t) = \left(t - \left(1 - \frac{\sqrt{2}}{2}\right)\right) \cdot g(t) = tg(t) - \left(1 - \frac{\sqrt{2}}{2}\right)g(t)$$

on the interval $t \in \left[0, 1 + \frac{\sqrt{2}}{2}\right]$. Note that the zeros of the function h are $t = 1 \pm \frac{\sqrt{2}}{2}$, while outside these zeros the inequality $h(t) < 0$ holds. Now we compute

$$g(x) + g(y) + g(z) + g(w)$$

$$= (x^2 + y^2 + z^2 + w^2) - 2 \cdot (x + y + z + w) + \left(\frac{1}{2} \cdot 4\right)$$

$$= 6 - 2 \cdot 4 + 2 = 0,$$

$$xg(x) + yg(y) + zg(z) + wg(w)$$

$$= (x^3 + y^3 + z^3 + w^3) - 2 \cdot (x^2 + y^2 + z^2 + w^2) + \frac{1}{2} \cdot (x + y + z + w)$$

$$= 10 - 2 \cdot 6 + \frac{1}{2} \cdot 4 = 0.$$

Therefore

$$h(x) + h(y) + h(z) + h(w)$$

$$= (xg(x) + yg(y) + zg(z) + wg(w)) - \left(1 - \frac{\sqrt{2}}{2}\right) (g(x) + g(y) + g(z) + g(w))$$

$$= 0 - \left(1 - \frac{\sqrt{2}}{2}\right) \cdot 0 = 0.$$

Since $4x \geq x + y + z + w = 4$, we have $x \geq 1$. If we had $x < 1 + \frac{\sqrt{2}}{2}$, then from the above we would get $h(x) < 0$, $h(y) \leq 0$, $h(z) \leq 0$ and $h(w) \leq 0$, in which case $h(x) + h(y) + h(z) + h(w) < 0$, a contradiction.

Hence $x \geq 1 + \frac{\sqrt{2}}{2}$. The value $1 + \frac{\sqrt{2}}{2}$ is attainable if we choose $(x, y, z, w) = \left(1 + \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}\right)$.

Remark: All solutions are of the form $x = 1 + \cos t$, $y = 1 + \sin t$, $z = 1 - \sin t$, $w = 1 - \cos t$, where $t \in \left[0, \frac{\pi}{4}\right]$. This is easy to verify in the context of Solution 1 by substituting $a = \cos t$, using the equality $a^2 + b^2 = 1$ and noticing that $\{c, d\} = \{-a, -b\}$.

O23 (*Seniors.*) In a trapezoid $ABCD$, the base AD is longer than the base BC , and $\angle DAB > \angle CDA$. The bisector of the interior angle at vertex A intersects the leg CD at point E , where $\angle EDA > \angle DAE$ and $\angle AED > 90^\circ$. Point F is chosen on the line BE so that $DF \parallel AB$. Point G is chosen on the circumcircle of triangle DEF so that $GE \parallel BC$ ($G \neq E$). Prove that the segments FG and CE have equal length.

Solution 1: We first notice that $\angle FDG = \angle FEG = \angle FBC = \angle EBC$ and $\angle DGF = \angle DEF = \angle CEB$ (Fig. 28). Hence triangles DGF and BEC are similar.

Since FG and CE are corresponding sides of these triangles, in order to prove their equality it is enough to prove the equality of any pair of corresponding sides. We show below that DF and BC are equal. For this, let P

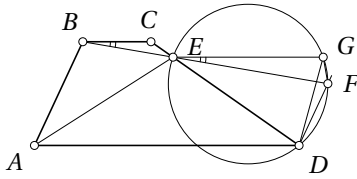


Fig. 28

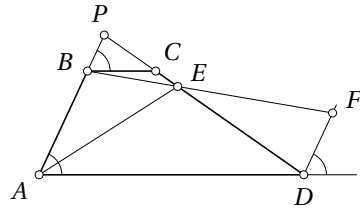


Fig. 29

be the intersection point of the extensions of the legs of trapezoid $ABCD$ (Fig. 29). Since $\angle PEB = \angle DEF$ and $\angle BPE = \angle FDE$, triangles PEB and DEF are similar. Therefore $\frac{DF}{BP} = \frac{DE}{PE}$, i.e.,

$$DF = \frac{DE}{PE} \cdot BP. \quad (24)$$

On the other hand, triangles CBP and DAP are also similar because their corresponding sides are parallel. Therefore $\frac{BC}{AD} = \frac{BP}{AP}$, that is,

$$BC = \frac{AD}{AP} \cdot BP. \quad (25)$$

But by the angle bisector theorem, $\frac{DE}{PE} = \frac{AD}{AP}$. Thus from Equalities (24) and (25) we get $DF = BC$.

Solution 2: As in Solution 1, we show that triangles DGF and BEC are similar. Since FG and CE are corresponding sides of these triangles, in order to prove their equality it is enough to prove the equality of any pair of corresponding sides. We show that DG and BE are equal. For this, let K be the second intersection point of the circumcircle of triangle DEF with the line AD (Fig. 30). Since EG and KD are parallel chords of the same circle, $EGDK$ is an isosceles trapezoid, so $DG = EK$. Hence, to solve the problem, it is enough to show that $BE = EK$.

For this, we first show that the points A, B, E and K are concyclic. Indeed, $\angle EKA = 180^\circ - \angle DKE = \angle EFD = \angle BFD = 180^\circ - \angle ABF = 180^\circ - \angle ABE$.

To show that $BE = EK$, it is enough to establish the equality of angles $\angle EKB = \angle KBE$. Since, by the above, $ABEK$ is a cyclic quadrilateral, we have $\angle EKB = \angle EAB = \angle DAE = \angle KAE = \angle KBE$, as required.

Remark: The statement of the problem also holds in the case where the bisector of the interior angle at vertex A intersects the extension of the leg CD ,

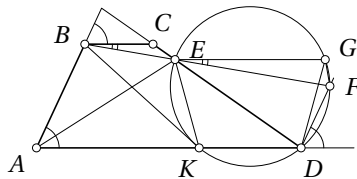


Fig. 30

or where the angle size relations are not as stated in the text. In general, the same proofs work, but they have to be adapted to the cases where the order of the vertices on the circumcircle of triangle DEF is D, E, F, G or D, G, E, F . In Solution 2, one also has to cover the special cases where AD is tangent to the circumcircle of triangle DEF or to the circumcircle of triangle ABE .

Selected Problems from the Final Round of National Olympiad

F1 (Grade 7.) Find the least two natural numbers n for which there exist prime numbers p and q such that both $n + 18$ and $4n + 7$ are divisible by pq .
Answer: 47, 112.

Solution: Since $n + 18$ is divisible by pq , $4n + 72$ is also divisible by pq . As $4n + 7$ is also divisible by pq , the difference $(4n + 72) - (4n + 7) = 65$ is divisible by pq . Since $65 = 5 \cdot 13$, the only possibility is $pq = 65$.

The two smallest natural numbers divisible by 65 are 65 and 130. From equations $n + 18 = 65$ and $n + 18 = 130$ we get $n = 47$ and $n = 112$. In both cases $4n + 7$ is also divisible by 65.

F2 (Grade 7.) Points A, B, C, D, E, F, G, H are marked on a line in this order. The length of segment AH is 200 cm, the length of segment BG is 85 cm and the length of segment CF is 60 cm. Considering all segments whose endpoints are two marked points, the sum of the lengths of the segments under consideration is 2026 cm. Find the length of segment DE .

Answer: 21 cm.

Solution 1: Every segment whose endpoints are two marked points consists of segments whose endpoints are two *consecutive* marked points (Fig. 31). We find out, parts of how many different segments are segments AB, BC, CD, DE, EF, FG and GH .

- Segment AB is a part of exactly 7 segments whose endpoints are two marked points ($AB, AC, AD, AE, AF, AG, AH$). Likewise, segment GH is a part of exactly 7 segments.
- Segment BC is a part of exactly 12 segments whose endpoints are two marked points, because to choose the first endpoint there are 2 possibilities and to choose the second endpoint there are 6 possibilities. Segment FG is also a part of exactly 12 segments.
- Similarly segments CD and EF are each a part of exactly 15 segments, and segment DE is a part of exactly 16 segments.



Fig. 31

By the conditions of the problem and the observations made, we get

$$\begin{aligned}
 2026 \text{ cm} &= 7AB + 12BC + 15CD + 16DE + 15EF + 12FG + 7GH \\
 &= 7(AB + BC + CD + DE + EF + FG + GH) \\
 &\quad + 5(BC + CD + DE + EF + FG) + 3(CD + DE + EF) + DE \\
 &= 7AH + 5BG + 3CF + DE.
 \end{aligned}$$

Hence $DE = 2026 \text{ cm} - 7AH - 5BG - 3CF$. Substituting $AH = 200 \text{ cm}$, $BG = 85 \text{ cm}$ and $CF = 60 \text{ cm}$ we get $DE = 21 \text{ cm}$.

Solution 2: Notice that if we move all marked points in segment BG to the left by x centimeters then the lengths of segments AH , BG , CF ja DE do not change. The sum of the lengths of all segments also remains the same.

Let us move all marked points in segment BG to the left so that points A and B coincide. Similarly move to the left all points in segment CF and finally the points D and E . Then the points A , B , C ja D coincide, and all lengths of segments given in the problem remain the same. Hence

$$\begin{aligned}
 AE &= BE = CE = DE, & AF &= BF = CF = DF, \\
 AG &= BG = CG = DG, & AH &= BH = CH = DH.
 \end{aligned}$$

In addition

$$\begin{aligned}
 EF &= DF - DE = CF - DE, & EG &= DG - DE = BG - DE, \\
 EH &= DH - DE = AH - DE, & FG &= CG - CF = BG - CF, \\
 FH &= CH - CF = AH - CF, & GH &= AH - AG = AH - BG.
 \end{aligned}$$

Consequently

$$\begin{aligned}
 2026 \text{ cm} &= 4DE + 4CF + 4BG + 4AH \\
 &\quad + (CF - DE) + (BG - DE) + (AH - DE) \\
 &\quad + (BG - CF) + (AH - CF) + (AH - BG).
 \end{aligned}$$

Here all lengths but DE are given. Solving the equation gives $DE = 21 \text{ cm}$.

F3 (Grade 7.) Let the midpoint of segment DE be A . On the same side of line DE , points B and C are chosen such that triangles ABD and ACE are isosceles with bases AD and AE , respectively, and $\angle CAB = 90^\circ$. On line CA , point F is chosen such that $\angle FDB = 90^\circ$. Knowing that $\angle ABC = 20^\circ$, find the size of angle AFD .

Answer: 140° .

Solution 1: From triangle ABC we get

$$\angle BCA = 180^\circ - \angle CAB - \angle ABC = 180^\circ - 90^\circ - 20^\circ = 70^\circ$$

(Fig. 32). Denote $\alpha = \angle EAC = \angle DAF$. Since triangle ACE is isosceles, also $\angle CEA = \alpha$. At the same time

$$\angle BAD = 180^\circ - \angle EAC - \angle CAB = 180^\circ - \alpha - 90^\circ = 90^\circ - \alpha,$$

and since triangle ABD is isosceles, $\angle ADB = 90^\circ - \alpha$. Therefore $\angle FDA = \angle FDB - \angle ADB = 90^\circ - (90^\circ - \alpha) = \alpha$, so triangle AFD is isosceles with the base AD . Since $\angle DAF = \alpha = \angle EAC$ and $\angle FDA = \alpha = \angle CEA$ and

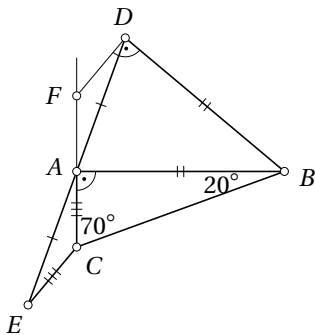


Fig. 32

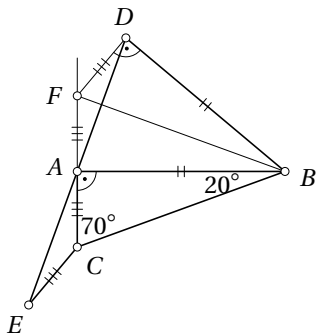


Fig. 33

also $AD = AE$, triangles ACE and AFD are equal. So $AF = AC$. Hence the sides AF and AC of triangles ABF and ABC are equal and the side AB is common, whereas $\angle BAF = 90^\circ = \angle CAB$ (Fig. 33). Thus triangles ABF and ABC are equal, whence $\angle AFB = \angle BCA = 70^\circ$. Also the sides DB and AB of triangles DBF and ABC are equal, DF and AC are equal and $\angle FDB = 90^\circ = \angle CAB$. Therefore triangles DBF and ABC are equal, whence $\angle BFD = \angle BCA = 70^\circ$. Finally

$$\angle AFD = \angle BFD + \angle AFB = 70^\circ + 70^\circ = 140^\circ.$$

Solution 2: Let G be the intersection point of lines BD and EC (Fig. 34). Since triangles ABD and ACE are isosceles, we get $\angle EDG = \angle BAD$ and $\angle GED = \angle EAC$. Thus $\angle GED + \angle EDG = \angle EAC + \angle BAD = 90^\circ$. From triangle DEG we now get $\angle DGE = 180^\circ - 90^\circ = 90^\circ$.

Since $\angle CGB = \angle EGB = 90^\circ = \angle CAB$, points C, A, G and B lie on the same circle (with diameter BC). From inscribed angles subtending arc AC we find $\angle AGC = \angle ABC = 20^\circ$. At the same time A is the center of the hypotenuse of the right triangle DEG , so it is the center of the circumcircle of the triangle, hence $AG = AD = AE$.

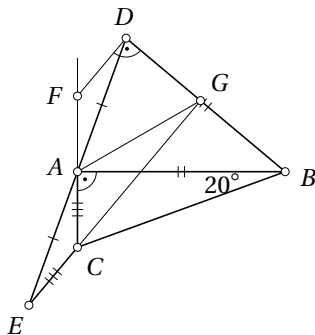


Fig. 34

From the isosceles triangle AEG we get

$$\angle CEA = \angle GEA = \angle AGE = \angle AGC = 20^\circ,$$

from the isosceles triangle ACE we get $\angle EAC = \angle CEA = 20^\circ$ and from vertex angles $\angle DAF = \angle EAC = 20^\circ$. On the other hand

$$\angle DGA = \angle DGE - \angle AGE = 90^\circ - 20^\circ = 70^\circ,$$

thus from the isosceles triangle ADG we get $\angle GAD = 180^\circ - 2 \cdot 70^\circ = 40^\circ$. Now $\angle GAF = \angle GAD + \angle DAF = 40^\circ + 20^\circ = 60^\circ$. Finally the sum of interior angles of quadrilateral $AFDG$ gives

$$\angle AFD = 360^\circ - \angle FDG - \angle DGA - \angle GAF = 360^\circ - 90^\circ - 70^\circ - 60^\circ = 140^\circ.$$

F4 (Grade 7.) There are two types of sticks of the same thickness in the warehouse: blue sticks are 14 cm long and red sticks are x cm long. For transportation, the sticks are arranged lengthwise in rows as short as possible so that each row contains only sticks of one colour and all rows are the same length. Later, all the sticks are cut into pieces as long as possible such that all the pieces are the same length. It is known that from each row 12 pieces of sticks can be made. Find all the possibilities that x can be.

Answer: 168, $\frac{56}{3}$, $\frac{21}{2}$ ja $\frac{7}{6}$.

Solution: Choose the unit so that the length of one piece is 1 unit. Let the lengths of blue and red sticks be a and b units correspondingly. Then a and b are positive integers with $\gcd(a, b) = 1$. Since the length of row is 12 units, a and b are divisors of 12. Also $\text{lcm}(a, b) = 12$, because the rows are the shortest possible. Hence $\gcd(a, b) \cdot \text{lcm}(a, b) = 12$, but because $\gcd(a, b) \cdot \text{lcm}(a, b) = ab$, we have $ab = 12$.

There are 6 ways to represent 12 as the product of two positive integers:

$$12 = 1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4 = 4 \cdot 3 = 6 \cdot 2 = 12 \cdot 1.$$

Options $a = 2, b = 6$ and $a = 6, b = 2$ do not meet the conditions, because $\gcd(a, b) \neq 1$. Consider the remaining 4 cases.

- If $a = 1, b = 12$ then 1 unit equals 14 cm, so 12 units equals 168 cm. Hence $x = 168$.
- If $a = 3, b = 4$ then 3 units equals 14 cm, so 4 units equals $\frac{4}{3} \cdot 14$ cm. Hence $x = \frac{56}{3}$.
- If $a = 4, b = 3$ then 4 units equals 14 cm, so 3 units equals $\frac{3}{4} \cdot 14$ cm. Hence $x = \frac{21}{2}$.
- If $a = 12, b = 1$ then 12 units equals 14 cm, so 1 unit equals $\frac{1}{12} \cdot 14$ cm. Hence $x = \frac{7}{6}$.

F5 (Grade 8.) Let m and n be integers satisfying

$$2m^2 - 3n^2 + m^2n^2 = 57.$$

Find all possible values of $m + n$.

Answer: 9, 5, -5, -9.

Solution 1: Subtracting 6 from both sides of the equation and factoring the left hand side we get

$$(m^2 - 3)(n^2 + 2) = 51.$$

Because $n^2 + 2 > 0$, also $m^2 - 3 > 0$. The number 51 can be represented as a product of two positive integers as $51 = 1 \cdot 51 = 3 \cdot 17 = 17 \cdot 3 = 51 \cdot 1$. Consider all cases.

- If $m^2 - 3 = 1$ and $n^2 + 2 = 51$ then $m^2 = 4$ and $n^2 = 49$. Hence $m = \pm 2$ and $n = \pm 7$. All possible values of $m + n$ are 9, -5, 5 and -9.
- If $m^2 - 3 = 3$ and $n^2 + 2 = 17$ then $m^2 = 6$ and $n^2 = 15$, so neither m nor n is an integer.
- If $m^2 - 3 = 17$ and $n^2 + 2 = 3$ then $m^2 = 20$, hence m is not an integer.
- The case $m^2 - 3 = 51$, $n^2 + 2 = 1$ is also impossible, because $n^2 \neq -1$.

Hence all possible values of $m + n$ are 9, -5, 5 and -9.

Solution 2: The equation is equivalent to $m^2 n^2 + 2m^2 = 3n^2 + 57$, where factoring both sides gives

$$m^2(n^2 + 2) = 3(n^2 + 19).$$

Since $n^2 + 2 > 0$ we can divide by $n^2 + 2$, so

$$m^2 = \frac{3(n^2 + 19)}{n^2 + 2}. \quad (26)$$

Let $\gcd(n^2 + 2, n^2 + 19) = d$. Then d divides $(n^2 + 19) - (n^2 + 2) = 17$, so $d = 1$ or $d = 17$.

If $d = 1$ then $n^2 + 2$ must be a divisor of 3, otherwise Eq. (26) cannot have integer solutions. Since $n^2 + 2 \geq 2$, the only possibility is $n^2 + 2 = 3$, whence $n^2 + 19 = 20$. Equation (26) now gives $m^2 = 20$, which does not have integer solutions.

If $d = 17$ then $n^2 + 2 = 17k$, where k is a positive integer. Substituting this into Eq. (26) gives

$$m^2 = \frac{3(k+1)}{k}.$$

Since $\gcd(k, k+1) = 1$, k must be a divisor of 3, so $k = 1$ or $k = 3$. If $k = 1$ then $m^2 = 6$, which does not have integer solutions. If $k = 3$ then $m^2 = 4$. From $n^2 + 2 = 17k = 17 \cdot 3 = 51$ we get $n^2 = 49$. Hence $m = \pm 2$ and $n = \pm 7$. All possible values of $m + n$ are 9, -5, 5 and -9.

F6 (Grade 8.) The lengths of sides AD ja CD of parallelogram $ABCD$ are correspondingly 4 cm and 9 cm. The bisector of angle BCD intersects lines AB and AD at points L and M . On line BC point E is chosen so that $\angle ELC = 90^\circ$. How many times is the area of parallelogram $ABCD$ bigger than the area of triangle ELM ?

Answer: $\frac{9}{5}$ or 1.8.

Solution: Denote by S_K the area of figure K .

A diagonal divides a parallelogram into two equal triangles, hence

$$S_{ABCD} = 2S_{ABC}. \quad (27)$$

From conditions of the problem $\angle DCL = \angle LCB$ (Fig. 35). Since $AB \parallel CD$, we have $\angle DCL = \angle BLC$. Hence $\angle LCB = \angle BLC$, so triangle LBC is isosceles with the vertex angle at B . Therefore $LB = BC = AD = 4$ cm. At the same time $AB = CD = 9$ cm. Triangles ABC ja LBC have a common altitude drawn from vertex C and the ratio of lengths of the corresponding bases AB and LB is $\frac{9}{4}$. Consequently

$$S_{ABC} = \frac{9}{4}S_{LBC}. \quad (28)$$

Notice that the equalities $\angle ELB = \angle ELC - \angle BLC = 90^\circ - \angle BLC$ and $\angle BEL = \angle CEL = 180^\circ - \angle ELC - \angle LCE = 90^\circ - \angle LCE$ hold. Since $\angle BLC = \angle LCB = \angle LCE$, we have $\angle ELB = \angle BEL$. Hence $BE = BL$, implying $EC = 2BC$. Thus triangles LBC ja LEC have a common altitude drawn from vertex L , but the ratio of lengths of the corresponding bases BC and EC is $\frac{1}{2}$. Therefore

$$S_{LBC} = \frac{1}{2}S_{LEC}. \quad (29)$$

Finally notice that since $AD \parallel BC$, we have $\angle LMA = \angle LCB$, and from vertex angles $\angle ALM = \angle BLC$. Hence triangles LBC and LAM are similar, whence $\frac{LM}{LC} = \frac{LA}{LB} = \frac{AB-LB}{LB} = \frac{AB}{LB} - 1 = \frac{9}{4} - 1 = \frac{5}{4}$. Thus triangles LEC and LEM have a common altitude drawn from vertex E , but the ratio of lengths of the corresponding bases LC and LM is $\frac{4}{5}$. Therefore

$$S_{LEC} = \frac{4}{5}S_{LEM}. \quad (30)$$

From Equations (27), (28), (29) and (30) we get

$$S_{ABCD} = 2S_{ABC} = 2 \cdot \frac{9}{4}S_{LBC} = 2 \cdot \frac{9}{4} \cdot \frac{1}{2}S_{LEC} = 2 \cdot \frac{9}{4} \cdot \frac{1}{2} \cdot \frac{4}{5}S_{LEM} = \frac{9}{5}S_{LEM}.$$

Consequently the area of parallelogram $ABCD$ is $\frac{9}{5}$ times bigger than the area of triangle ELM .

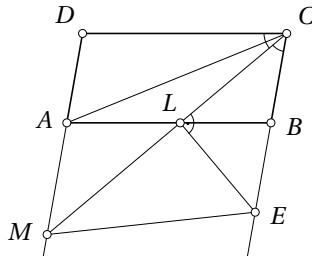


Fig. 35

F7 (Grade 8.) A 10×10 board is called a *sea*. A *ship* consists of at least 2 consecutive unit squares of the same row or column of the sea. Unit squares of different ships cannot coincide or have a common side, but a common vertex is allowed. Is it always true that all unit squares of the ships can be colored red or green so that no unit squares having a common side or vertex are of the same color?

Answer: No.

Solution 1: Denote the rows by numbers 1 to 10 and columns by letters A to J. Choose one ship consisting of 3 squares $B1 - C1 - D1$ and four ships consisting of 2 squares $A2 - A3$, $B4 - B5$, $C3 - C4$ and $E2 - E3$ (Fig. 36). Consider the path

$$B1 \rightarrow C1 \rightarrow D1 \rightarrow E2 \rightarrow D3 \rightarrow C3 \rightarrow B4 \rightarrow A3 \rightarrow A2 \rightarrow B1$$

(on Fig. 36 marked with dots). All squares belong to some ship and every two consecutive squares on the path have a common side or vertex. Hence all consecutive squares must have different colors. But the path has 10 squares, hence the first and the last squares must be of different colors, but both are the same square. Hence the required coloring is not possible.

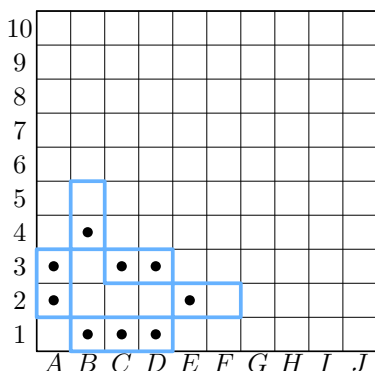


Fig. 36

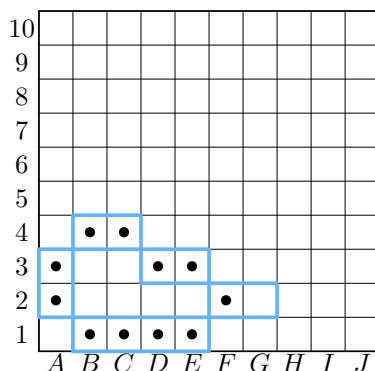


Fig. 37

Solution 2: Denote the rows and columns similarly as in Solution 1. Choose one ship consisting of 4 squares $B1 - C1 - D1 - E1$ and four ships consisting of 2 squares $A2 - A3$, $B4 - C4$, $D3 - E3$ and $F2 - G2$ (Fig. 37). Consider the path

$$B1 \rightarrow C1 \rightarrow D1 \rightarrow E1 \rightarrow F2 \rightarrow E3 \rightarrow D3 \rightarrow C4 \rightarrow B4 \rightarrow A3 \rightarrow A2 \rightarrow B1$$

(on Fig. 37 marked with dots). Since the path has 12 squares we get a contradiction as in solution 1. Hence the required coloring is not possible.

Remark: Solution 2 shows that the answer is no, even if all ships must consist of even number of squares.

F8 (Grade 8.) Let a , b and c be positive integers satisfying the conditions

$5a - 2b = 20c$ and $a + 4b = 26c$. How many pairs of positive integers (m, n) exist such that $ma + nb = 2026c$?

Answer: 68.

Solution: Multiplying equation $5a - 2b = 20c$ by 2 and adding equation $a + 4b = 26c$ we get $11a = 66c$, thus $a = 6c$. Multiplying $a + 4b = 26c$ by 5 and subtracting $5a - 2b = 20c$ we get $22b = 110c$, thus $b = 5c$.

Equation $ma + nb = 2026c$ now takes the form $m \cdot 6c + n \cdot 5c = 2026c$, hence

$$6m + 5n = 2026. \quad (31)$$

Now n must be even, so the last digit of $6m$ must be 6, hence the last digit of m must be 1 or 6. Therefore Eq. (31) holds if and only if the last digit of m is 1 or 6 and $6m < 2026$. Since $336 \cdot 6 = 2016 < 2026$, but $341 \cdot 6 = 2046 > 2026$, m can only be 1, 6, 11, 16, $\dots$, 331, 336 which makes 68 possibilities. Since for every m there exists a unique n , there are 68 pairs.

F9 (Grade 9.) Find all positive integers n such that the number $3^n - 1$ is the product of two consecutive positive integers.

Answer: 1.

Solution 1: Let n satisfy the condition of the problem, then for some positive integer m we have

$$3^n = m(m + 1) + 1. \quad (32)$$

The left hand side of this equation is divisible by 3. If $m = 3k$ or $m = 3k + 2$, then the right hand side is not divisible by 3, contradiction. Thus we can only have $m = 3k + 1$. Substituting it into Eq. (32) yields

$$3^n = (3k + 1)(3k + 2) + 1 = 9k^2 + 9k + 3 = 3(3k^2 + 3k + 1).$$

The number $3k^2 + 3k + 1 = 3k(k + 1) + 1$ is never divisible by 3, whence $3(3k^2 + 3k + 1)$ is not divisible by 9. But if $n > 1$, then 3^n is divisible by 9, contradiction. The case $n = 1$ works for $m = 1$.

Solution 2: Let n satisfy the condition of the problem, then for some positive integer m we have

$$m^2 + m - (3^n - 1) = 0.$$

Consider this as a quadratic equation with respect to m . The discriminant of the equation is $1 + 4(3^n - 1) = 4 \cdot 3^n - 3 = 3(4 \cdot 3^{n-1} - 1)$. For an integer solution m to exist, the discriminant must be a perfect square, which yields

$$3(4 \cdot 3^{n-1} - 1) = x^2 \quad (33)$$

for some integer x . The left hand side of Eq. (33) is divisible by 3, so the right hand side must be divisible by 3 as well. This means that x is divisible by 3, but then x^2 is divisible by 9 as well. But for the left hand side to be divisible by 9, the factor $4 \cdot 3^{n-1} - 1$ must be divisible by 3. If $n > 1$, then 3^{n-1} is divisible by 3 so $4 \cdot 3^{n-1} - 1$ is not divisible by 3, contradiction. The case $n = 1$ works for $m = 1$.

F10 (Grade 9.) Triangle ABC has a right-angle at vertex A . The point D on the side BC is chosen such that the segments BA and BD are of equal length. Let E be the reflection of the point A across the line BC . The rays BE and AD intersect in point F . Triangle ACF has circumcenter O . Prove that points B, C, F , and O are concyclic.

Solution: Denote $\alpha = \angle DAC$ (Fig. 38). Then

$$\angle BAD = \angle BAC - \angle DAC = 90^\circ - \alpha$$

and since BA and BD are equal, we also have

$$\angle CBA = \angle DBA = 180^\circ - 2(90^\circ - \alpha) = 2\alpha.$$

By the choice of point E we get $\angle EBC = \angle CBA = 2\alpha$. From the circumcircle of ACF we get $\angle FOC = 2\angle FAC = 2\angle DAC = 2\alpha$. In conclusion $\angle FBC = \angle EBC = \angle FOC$, which yields that the points B, C, F and O are concyclic.

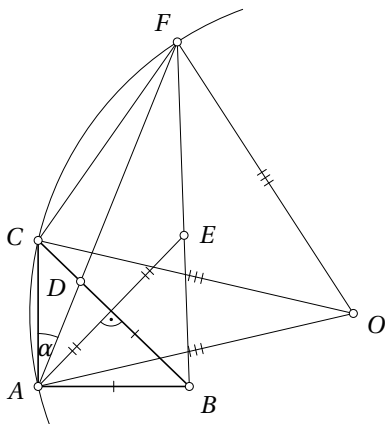


Fig. 38

F11 (Grade 9.) Ats and Pets play the following game on an 8×8 grid of unit squares. At the start all unit squares are uncoloured. On each move a player colours one uncoloured unit square red or blue that has a common side with at most one unit square of that colour. They take turns with Ats starting. The player who cannot make a move loses. Is it possible for either player to beat the opponent in every counterplay, and if so, which one?

Answer: Yes, Pets.

Solution: Pets' strategy is to take the square Ats coloured most recently, reflect it across the horizontal centerline of the grid, and colour it the opposite colour from Ats' square (Fig. 39).

We will show that this strategy always works. Call a game state *symmetric* if any two symmetric unit squares with respect to the horizontal centerline are either both uncoloured or both coloured, but with different colours. The initial state is clearly symmetric. If the game state is symmetric before Ats'

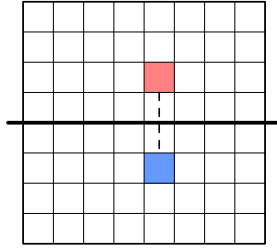


Fig. 39

turn, then Ats is forced to choose a square whose counterpart is also uncoloured. Furthermore, if Ats' chosen square has at most one neighbour of the same colour, then its symmetric counterpart has at most one neighbour of the other colour, so Pets is able to colour it with the other colour and put the game back into a symmetric state. Thus it must be Ats who eventually cannot make a move and loses.

F12 (Grade 9.) A point K is chosen on the side AB of the rectangle $ABCD$ and a point L is chosen on the side BC . The segment KL and the diagonal BD of the rectangle intersect in the point M . It is known that the lengths of the segments BM and MD are in the ratio $1 : 4$, while the lengths of the segments KM and ML are in the ratio $2 : 3$. What fraction is the area of triangle BLM of the area of rectangle $ABCD$?

Answer: $\frac{1}{20}$.

Solution: Let S_K denote the area of K .

Triangles BKM and BLM share a common altitude drawn from vertex B , and the corresponding bases have a ratio of lengths of $2 : 3$ (Fig. 40). Thus

$$S_{BKM} = \frac{2}{3} S_{BLM} \quad (34)$$

and therefore also

$$S_{BLK} = S_{BLM} + S_{BKM} = \frac{5}{3} S_{BLM}.$$

Triangles BLD and BLM share an altitude drawn from L , and the respective bases have a ratio of lengths of $\frac{BD}{BM} = \frac{BM+MD}{BM} = \frac{BM+4BM}{BM} = 5$. Thus

$$S_{BLD} = 5S_{BLM}.$$

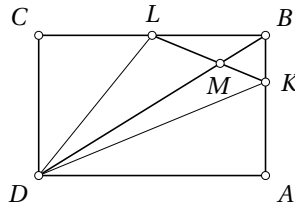


Fig. 40

Analogously $S_{BKD} = 5S_{BKM}$, which together with Eq. (34) gives

$$S_{BKD} = 5 \cdot \frac{2}{3} S_{BLM} = \frac{10}{3} S_{BLM}.$$

Triangles BLK and BKD share the base BK , whereas their areas have a ratio of $\frac{5}{3} : \frac{10}{3} = 1 : 2$. Thus the corresponding altitudes BL and BC have a ratio of lengths of $1 : 2$. Triangles BLD and BCD share the altitude CD , whereas the corresponding bases have a ratio of lengths of $1 : 2$. Thus

$$S_{BCD} = 2S_{BLD}.$$

On the other hand $S_{ABCD} = 2S_{BCD}$. In conclusion

$$S_{ABCD} = 2S_{BCD} = 2 \cdot 2S_{BLD} = 2 \cdot 2 \cdot 5S_{BLM} = 20S_{BLM}.$$

This means that the area of BLM is $\frac{1}{20}$ of the area of $ABCD$.

F13 (Grade 10.) Find all pairs (a, b) of positive integers such that both $a^2 - 4b$ and $b^2 - 4a$ are perfect squares.

Answer: $(4, 4)$, $(5, 6)$, $(6, 5)$.

Solution 1: Let (a, b) satisfy the conditions of the problem and let $a \leq b$ without loss of generality. Since $b^2 - 4a < b^2$ and the numbers b^2 and $b^2 - 4a$ have the same parity, we must have $b^2 - 4a \leq (b - 2)^2 = b^2 - 4b + 4$. Cancelling common terms and dividing by 4 yields $b \leq a + 1$. Together with the assumption $a \leq b$, we are left with two cases: $b = a$ and $b = a + 1$.

- Let $b = a$. Then $a^2 - 4a$ is a perfect square. But as $a^2 - 4a < (a - 2)^2$ and the numbers a^2 and $a^2 - 4a$ have the same parity, we must have $a^2 - 4a \leq (a - 4)^2 = a^2 - 8a + 16$, from which $a \leq 4$. For $0 < a < 4$ we have $a^2 - 4a < 0$, contradiction. For $a = 4$ we have $a^2 - 4a = 0 = 0^2$, which yields the solution $(4, 4)$.
- Let $b = a + 1$. Then $a^2 - 4(a + 1) = a^2 - 4a - 4$ is a perfect square. As $a^2 - 4a - 4 < (a - 2)^2$ and the numbers a^2 and $a^2 - 4a - 4$ have the same parity, we must have $a^2 - 4a - 4 \leq (a - 4)^2 = a^2 - 8a + 16$, from which $a \leq 5$. For $0 < a < 5$ we have $a^2 - 4(a + 1) < 0$, contradiction. For $a = 5$ we have $a^2 - 4(a + 1) = 1 = 1^2$, which yields the solution $(5, 6)$. Both this and $(6, 5)$ satisfy the conditions of the problem.

Solution 2: The expression $a^2 - 4b$ being a perfect square is equivalent to the quadratic equation $x^2 - ax + b$ having integer solutions x_1 and x_2 . Vieta's formulas yield $a = x_1 + x_2$ and $b = x_1 x_2$ which in turn give

$$b - a + 1 = x_1 x_2 - x_1 - x_2 + 1 = (x_1 - 1)(x_2 - 1). \quad (35)$$

For $x_1 + x_2$ to be positive, at least one of x_1 and x_2 must be positive, and then for $x_1 x_2$ to be positive, both x_1 and x_2 must be positive.

Analogously we find that there exist positive integers y_1 and y_2 such that

$$a - b + 1 = y_1 y_2 - y_1 - y_2 + 1 = (y_1 - 1)(y_2 - 1). \quad (36)$$

Adding (35) and (36) yields $2 = (x_1 - 1)(x_2 - 1) + (y_1 - 1)(y_2 - 1)$. As x_1, x_2, y_1, y_2 are positive integers, we are left with only three cases.

- If $(x_1 - 1)(x_2 - 1) = 2$ and $(y_1 - 1)(y_2 - 1) = 0$, then x_1 and x_2 are 2 and 3 in some order. Then $a = 2 + 3 = 5$ and $b = 2 \cdot 3 = 6$.
- If $(x_1 - 1)(x_2 - 1) = 1$ and $(y_1 - 1)(y_2 - 1) = 1$, then $x_1 = x_2 = 2$, from which $a = b = 2 + 2 = 4$.
- The case $(x_1 - 1)(x_2 - 1) = 0$ and $(y_1 - 1)(y_2 - 1) = 2$, is symmetric with the first case and yields $a = 6, b = 5$.

It is easy to see that all the pairs found meet the conditions of the problem.

F14 (Grade 10.) Find all triples (x, y, z) of positive real numbers which satisfy the system of equations

$$\begin{cases} x^{\frac{1}{y}} = z, \\ y^{\frac{1}{z}} = x, \\ z^{\frac{1}{x}} = y. \end{cases}$$

Answer: $(1, 1, 1)$.

Solution 1: Notice that $x = y = z = 1$ works. As the system of equations is cyclically symmetric, assume now without loss of generality that $x \neq 1$.

If $x > 1$, then due to $\frac{1}{y} > 0$ the first equation yields $z > 1$. Due to $\frac{1}{x} > 0$ the third equation yields $y > 1$. So all of x, y, z are greater than 1. Assume without loss of generality that x is the smallest of them. But then due to $\frac{1}{y} < 1$ we have $z = x^{\frac{1}{y}} < x$, contradiction.

Analogously in the case $x < 1$ we find that all of x, y, z are less than 1. Assume without loss of generality that x is the smallest of them. But then due to $\frac{1}{y} > 1$, we have $z = x^{\frac{1}{y}} < x$, contradiction.

Solution 2: The given system is equivalent to

$$\begin{cases} x = z^y, \\ y = x^z, \\ z = y^x. \end{cases} \quad (37)$$

Substituting y from the second equation to the third yields

$$z = (x^z)^x = x^{zx}. \quad (38)$$

Plugging this into the first equation of (37) yields $x = (x^{zx})^y = x^{xyz}$. Hence $x = 1$ or $xyz = 1$. But combining Eq. (38) with the second equation of (37) gives $xyz = x \cdot x^z \cdot x^{zx} = x^{1+z+zx}$. Thus $x^{1+z+zx} = 1$. As $1 + z + zx > 0$, we conclude $x = 1$. From the system (37), we now get also $y = 1$ and $z = 1$.

F15 (Grade 10.) In triangle ABC , the circle centered at A and passing through C also passes through the midpoint of BC . The circle centered at B and passing through C , also passes through the midpoint of AC . Prove that ABC is isosceles.

Solution: Let the midpoints of BC and AC be D and E respectively. Then $AC = AD$ and $BC = BE$, so the triangles ACD and BCE are isosceles

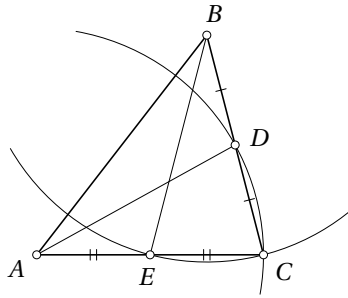
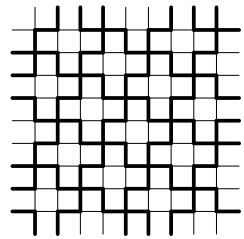


Fig. 41

(Fig. 41). At the same time they share a base angle at the vertex C , so they are similar. This yields the equation $\frac{AC}{BC} = \frac{CD}{CE}$. From the definitions of D and E we get $CD = \frac{1}{2}BC$ and $CE = \frac{1}{2}AC$, so $\frac{CD}{CE} = \frac{\frac{1}{2}BC}{\frac{1}{2}AC} = \frac{BC}{AC}$. In conclusion $\frac{AC}{BC} = \frac{BC}{AC}$. Rearranging this gives $AC^2 = BC^2$, which in turn yields $AC = BC$. Thus ABC is isosceles, as desired.

F16 (Grade 10.) The coordinate plane is divided into crosses consisting of 5 unit squares such that each unit square is contained in exactly one cross. Find all pairs (n, m) of positive integers, for which a rectangle with dimensions $n \times m$ can be placed onto the grid such that its sides overlap with the grid lines and no cross has exactly one unit square covered by the rectangle.



Answer: $(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2), (4, 4)$.

Solution: The rectangle covers exactly one unit square of a cross if and only if the central unit square of the cross shares an external side with the rectangle (Fig. 42). Thus we need to find the pairs (n, m) , for which there exists an $n \times m$ rectangle, which does not share an external side with any central unit square of any cross in the grid.

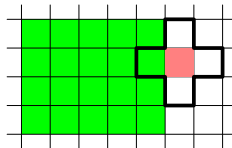


Fig. 42

Consider the line that coincides with the upper side of the rectangle. We number the unit squares of the row directly above the line periodically with the numbers 1, 2, 3, 4, 5 (Fig. 43). Then we see that all unit squares in that row which are central in their cross have the same number, and that no

Answer: No.

Solution: Let $n = p^5$ for a prime p . The divisors of n are $1, p, p^2, p^3, p^4, p^5$ and their sum is $1 + p + p^2 + p^3 + p^4 + p^5 = (1 + p^3)(1 + p + p^2)$.

We will show that $1 + p^3$ and $1 + p + p^2$ are coprime. Assume that $d \mid 1 + p^3$ and $d \mid 1 + p + p^2$. Then $d \mid p(1 + p + p^2) - (1 + p^3) = p^2 + p - 1$ and thus also $d \mid (p^2 + p + 1) - (p^2 + p - 1) = 2$. As $p + p^2 = p(p + 1)$ is even, $1 + p + p^2$ must be odd, so $d = 2$ is not possible. Thus we can only have $d = 1$, which means that $1 + p^3$ and $1 + p + p^2$ are coprime.

Thus, for $(1 + p^3)(1 + p + p^2)$ to be a perfect square, both factors $1 + p^3$ and $1 + p + p^2$ would have to be perfect squares themselves. This is impossible though, as $p^2 < 1 + p + p^2 < 1 + 2p + p^2 = (p + 1)^2$, so $1 + p + p^2$ is between two consecutive perfect squares.

F18 (Grade 11.) Let n be a positive integer. Order the numbers $(n!)^n$, $n^{n!}$ and $(n^n)!$ by size and find all equality cases.

Answer: $(n!)^n \leq n^{n!} \leq (n^n)!$, where the first inequality is an equality for $n = 1$ and $n = 2$ and the second inequality is an equality for $n = 1$.

Solution 1: Throughout the solution, we will use the inequality

$$n! = 1 \cdot 2 \cdot \dots \cdot n = 2 \cdot \dots \cdot n \leq \underbrace{n \cdot \dots \cdot n}_{n-1 \text{ factors}} = n^{n-1},$$

where for $n \geq 3$ the inequality is strict.

If $n = 1$ or $n = 2$, then $(n!)^n = n^{n!}$. In the case $n \geq 3$ however

$$(n!)^n < (n^{n-1})^n = n^{(n-1) \cdot n} \leq n^{n!}.$$

On the other hand

$$n^{n!} \leq n^{n^{n-1}} = \underbrace{n \cdot \dots \cdot n}_{n^{n-1} \text{ factors}}.$$

Comparing the first factor n in this product to the product $1 \cdot 2 \cdot \dots \cdot n$, the second factor to the product $(n + 1) \cdot (n + 2) \cdot \dots \cdot 2n$ etc., we get a comparison with $(n^n)!$. The comparison is a strict inequality for all $n \geq 2$. Thus

$$\underbrace{n \cdot \dots \cdot n}_{n^{n-1} \text{ factors}} \leq (n^n)!,$$

where equality holds only for $n = 1$. In conclusion $n^{n!} \leq (n^n)!$, where equality holds only for $n = 1$.

Thus we always have $(n!)^n \leq n^{n!} \leq (n^n)!$, where the first inequality is an equality for $n = 1$ and $n = 2$ and the second one is an equality for $n = 1$.

Solution 2: If $n = 1$ or $n = 2$, then $(n!)^n = n^{n!}$. We will show that for $n \geq 3$ we have $(n!)^n < n^{n!}$. Raising both sides to the power of $\frac{1}{n}$ yields the equivalent inequality $n! < n^{(n-1)!}$, where dividing by n yields the equivalent inequality $(n - 1)! < n^{(n-1)!-1}$. This is a special case of the inequality $m < n^{m-1}$, which holds for all $n \geq 3$ and $m \geq 2$. This can easily be proved

by induction over m . For $m = 2$ the inequality clearly holds and assuming the inequality to hold for some m and n , we get

$$n^{(m+1)-1} = n^{(m-1)+1} = n^{m-1} \cdot n > m \cdot n > m \cdot 2 = m + m > m + 1.$$

This completes the induction step, proving the desired inequality.

If $n = 1$, then $n^{n!} = 1 = (n^n)!$. We will show that $n^{n!} < (n^n)!$ for $n \geq 2$. For this notice that

$$\begin{aligned} (n^n)! &= (1 \cdot 2 \cdot \dots \cdot n) \cdot ((n+1) \cdot (n+2) \cdot \dots \cdot n^n) \\ &> (n+1) \cdot (n+2) \cdot \dots \cdot n^n > \underbrace{n \cdot n \cdot \dots \cdot n}_{n^n - n \text{ factors}} = n^{n^n - n}. \end{aligned}$$

Thus it remains to prove $n^{n!} \leq n^{n^n - n}$, which is equivalent to $n! \leq n^n - n$. We will prove it by induction. In the case $n = 2$, equality holds. Assuming $n! \leq n^n - n$ for some n , we get

$$\begin{aligned} (n+1)! &= (n+1) \cdot n! \leq (n+1) \cdot (n^n - n) \\ &< (n+1) \cdot (n^n - 1) = (n+1) \cdot n^n - (n+1) \\ &< (n+1) \cdot (n+1)^n - (n+1) = (n+1)^{n+1} - (n+1). \end{aligned}$$

This completes the induction step, proving the desired inequality.

Thus we always have $(n!)^n \leq n^{n!} \leq (n^n)!$, where the first inequality is an equality for $n = 1$ and $n = 2$ and the second one is an equality for $n = 1$.

F19 (Grade 11.) In an acute triangle ABC , the circumcentre is O and the orthocentre is H . The lines AH and BH intersect the circumcircle of ABC at $D \neq A$ and $E \neq B$. Let $K \neq C$ be the intersection of BC and the circumcircle of CEO . Let L be the circumcenter of BHD . Prove that OKL is isosceles.

Solution: We will show that $\angle OKL = \angle KLO$.

By definition, L lies on the perpendicular bisector of HD (Fig. 49). Using the circumcircle of ABC and the altitudes in ABC , we get

$$\angle DBC = \angle DAC = 90^\circ - \angle ACB = \angle CBH \quad (39)$$

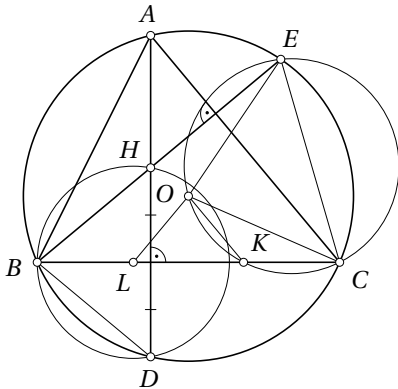


Fig. 49

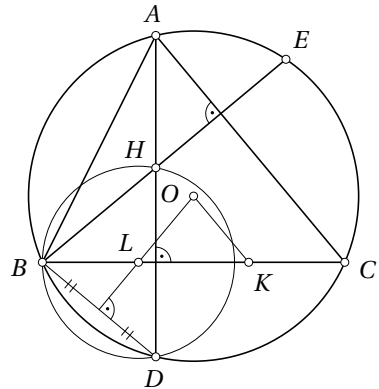


Fig. 50

and as $BC \perp HD$, the altitude and angle bisector drawn from B in triangle BHD coincide. This makes BHD isosceles and means that BC is also the perpendicular bisector of HD . Thus L lies on BC . From this and the cyclic quadrilateral $CKOE$ we get $\angle OKL = 180^\circ - \angle CKO = \angle OEC$. Since OC and OE are equal as radii, we get $\angle OEC = \frac{180^\circ - \angle COE}{2} = 90^\circ - \frac{\angle COE}{2}$. But from the circumcircle of ABC we get $\frac{\angle COE}{2} = \angle CBE = \angle CBH$. In conclusion

$$\angle OKL = 90^\circ - \angle CBH. \quad (40)$$

By definition, L also lies on the perpendicular bisector of BD (Fig. 50). As both B and D lie on the circumcircle of ABC , O lies on the perpendicular bisector of BD as well. Thus OL is the perpendicular bisector of BD . As L lies on BC , we get $\angle KLO = 90^\circ - \angle DBC$. Together with Eq. (39), we get

$$\angle KLO = 90^\circ - \angle CBH. \quad (41)$$

Finally, combining (40) and (41) yields $\angle OKL = \angle KLO$, as desired.

F20 (Grade 11.) A finite number of positive integers is written on the blackboard. On each turn, two numbers x and y such that $x < y$ are replaced with the numbers $x + \gcd(x+1, y+1) - 1$ and $y - \gcd(x, y)$. Prove that after finitely many turns, all numbers on the blackboard will be equal.

Solution: As $\gcd(x+1, y+1) \geq 1$ and $\gcd(x, y) < y$ for $x < y$, the new numbers written on each turn are positive. We will also show that both new numbers are smaller than y . Indeed, we have $y - \gcd(x, y) < y$ as $\gcd(x, y) > 0$, and since

$$\gcd(x+1, y+1) = \gcd(x+1, (y+1) - (x+1)) = \gcd(x+1, y-x) \leq y-x,$$

we also have $x + \gcd(x+1, y+1) - 1 \leq x + (y-x) - 1 = y-1 < y$.

Assume now for contradiction that for some starting position there exists an infinite sequence of turns. As the number of integers written on the board is fixed and we have shown that the greatest integer on the board cannot increase, there are a finite number of possible positions, so some position must be repeated. Let k be the greatest number erased between two instances of this position. But we have shown that only numbers less than k will be written on the board. Therefore there will never be enough copies of k on the board for the position to repeat. This gives the desired contradiction. The contradiction implies that we will reach a position with all numbers equal in a finite number of turns.

F21 (Grade 12.) Find all triples (x, y, z) of positive integers which satisfy the equation $x^{y^z} = y^{x^z}$.

Answer: $(2, 4, 1)$, $(4, 2, 1)$ and all triples (x, y, z) where $x = y$.

Solution: Any triple with $x = y$ clearly works. Since $x = 1$ implies $y = 1$ and vice versa, assume from now on that $1 < x < y$ without loss of generality.

Denote $x^z = t$ and $\frac{y^z}{x^z} = r$. Then $t \geq 2$ and $r > 1$. We obtain $y^z = rt$ and can rewrite the given equation as $x^{rt} = y^t$. This is equivalent to $x^r = y$.

Therefore $rt = y^z = (x^r)^z = (x^z)^r = t^r$, which can be rewritten as

$$t^{r-1} = r. \quad (42)$$

Raising the positive integer t to a positive rational power $r - 1$ yields either an integer or an irrational number as a result. But on the other hand the result is r , which is rational by definition. Thus r must be an integer.

Using $t \geq 2$, Eq. (42) yields $2^{r-1} \leq r$. Here equality holds for $r = 2$, whereas for $r \geq 3$ a simple induction shows that $2^{r-1} > r$. Thus we can only have solutions for $t = 2, r = 2$. From $x^z = t = 2$ the only solution is $x = 2, z = 1$ and $y = x^r = 2^2 = 4$. The triple with x and y swapped also works.

F22 (Grade 12.) Let a, b, c be real numbers from $[0, 1]$. Prove that

$$9abc + 9 \geq 2(a + b + c)^2.$$

Solution 1: Since $a, b, c \in [0, 1]$, we have $0 \leq ab \leq 1$ and $0 \leq c \leq 1$. This implies $(1 - ab)(1 - c) \geq 0$ which can be rewritten as $1 + abc \geq ab + c$. Analogously $1 + abc \geq bc + a$ and $1 + abc \geq ca + b$. Adding up those inequalities and multiplying by 3 yields

$$9abc + 9 \geq 3(a + b + c) + 3(ab + bc + ca). \quad (43)$$

Note that $a \geq a^2$ and $a \geq ab$, giving us $3a \geq 2a^2 + ab$. Analogously we have $3b \geq 2b^2 + bc$ and $3c \geq 2c^2 + ca$. Adding them up yields

$$3(a + b + c) \geq 2(a^2 + b^2 + c^2) + (ab + bc + ca).$$

Using this in (43), we get

$$9abc + 9 \geq 2(a^2 + b^2 + c^2) + 4(ab + bc + ca) = 2(a + b + c)^2.$$

Solution 2: Notice that if $a + b + c \leq 2$, then

$$9abc + 9 \geq 9 > 8 = 2 \cdot 2^2 \geq 2(a + b + c)^2.$$

Now assume $a + b + c \geq 2$. Define $x = 1 - a$, $y = 1 - b$ and $z = 1 - c$. Then x, y and z are also from $[0, 1]$. Substituting $a = 1 - x$, $b = 1 - y$ and $c = 1 - z$ into the desired inequality along with simplification yields

$$3(x + y + z) + 9(xy + xz + yz) \geq 2(x + y + z)^2 + 9xyz. \quad (44)$$

Note that $0 \leq x + y + z = 3 - (a + b + c) \leq 3 - 2 = 1$. Therefore, we have $3(x + y + z) \geq 2(x + y + z)^2$ and $9(xy + xz + yz) \geq 9xy \geq 9xyz$, so the inequality (44) holds.

Remark: This problem can also be approached using derivatives.

F23 (Grade 12.) A positive integer n is given. Find all positive integers that can be expressed as the sum of one or more distinct positive odd integers, each of which is less than $2n$.

Answer: All positive integers up to n^2 , except 2 and $n^2 - 2$.

Solution 1: Since $1 + 3 + \dots + (2n - 1) = n^2$, it is clearly impossible to achieve numbers greater than n^2 . The number 2 cannot be expressed as the

sum of distinct positive odd integers, and neither can $n^2 - 2$, as then the unused terms would have to add up to 2.

We will prove by induction on n that all positive integers up to n^2 except 2 and $n^2 - 2$ can be expressed as the sum of distinct positive odd integers. For $n = 1, 2, 3$ we can use the sums $1 = 1$, $3 = 3$, $4 = 1 + 3$, $5 = 5$, $6 = 1 + 5$, $8 = 3 + 5$ and $9 = 1 + 3 + 5$. Assume now that the claim holds for some $n \geq 3$. Then we can already achieve all numbers from 1 to n^2 except 2 and $n^2 - 2$. Thus it remains to express $n^2 - 2$ and the numbers from $n^2 + 1$ to $(n + 1)^2$, except $(n + 1)^2 - 2$. Using the newly available addend $2n + 1$, the remaining addends must be used to express $n^2 - 2n - 3$ and the numbers from $n^2 - 2n$ to $(n + 1)^2 - (2n + 1) = n^2$, except $(n + 1)^2 - 2 - (2n + 1) = n^2 - 2$, which can clearly be done by the induction assumption, as both $n^2 - 2n - 3$ and $n^2 - 2n$ are non-negative and less than $n^2 - 2$ for $n \geq 3$.

Solution 2: Like in Solution 1, notice that the numbers 2, $n^2 - 2$ and the numbers greater than n^2 cannot be expressed. We will show that the rest of the numbers can be expressed.

Let x be a positive integer satisfying $x \leq n^2$, $x \neq 2$ and $x \neq n^2 - 2$. We define the sequence of partial sums $a_n, a_{n-1}, \dots, a_0$ recursively, starting from $a_n = x$ and for each $k \leq n$

$$a_{k-1} = \begin{cases} a_k - (2k - 1), & \text{if } a_k \geq 2k - 1 \text{ and } a_k - (2k - 1) \neq 2, \\ a_k & \text{otherwise.} \end{cases}$$

We show by mathematical induction over k , that for all $k \leq n$ the following three conditions are satisfied: $0 \leq a_k \leq k^2$, $a_k \neq 2$ and $a_k \neq k^2 - 2$. The base case $k = n$ holds by definition. Assume now that for $k = m$ the conditions are satisfied, and we will show that they are also satisfied for $k = m - 1$.

First consider the case $a_{m-1} = a_m - (2m - 1)$. Then $0 \leq a_{m-1}$ and $a_{m-1} \neq 2$ and since $a_m \leq m^2$, also $0 \leq a_{m-1} \leq m^2 - (2m - 1) = (m - 1)^2$. As $a_m \neq m^2 - 2$, we also get $a_{m-1} \neq (m - 1)^2 - 2$. Thus all conditions are met. Second consider the case where $a_{m-1} = a_m$ due to $a_m - (2m - 1) = 2$. Then $a_m = 2m + 1$ and $a_{m-1} \neq 2$. Then also $2m + 1 \neq (m - 1)^2 - 2$, because the quadratic equation $m^2 - 4m - 2 = 0$ has no integer solutions m . The inequality $2m + 1 \leq (m - 1)^2$ is equivalent to $m(m - 4) \geq 0$ and holds for $m \geq 4$. For $m = 1, 2, 3$ we'd get $a_1 = 3 > 1 = 1^2$, $a_2 = 5 > 4 = 2^2$ and $a_3 = 7 > 3^2 - 2$ respectively, all of which are contradictory.

Finally consider the case where $a_{m-1} = a_m$ due to $a_m < 2m - 1$. Then $a_m \leq 2m - 2$ and $a_{m-1} \neq 2$. If $m \geq 4$, then analogously to the previous case we get $a_{m-1} < 2m - 1 \leq (m - 1)^2 - 2$. If $m = 3$, then $(m - 1)^2 - 2 = 2$, but $a_{m-1} \neq 2$. If $m = 2$ or $m = 1$, then $a_{m-1} \neq (m - 1)^2 - 2 < 0$. We finally note that $2m - 2 \leq (m - 1)^2$ is equivalent to $(m - 1)(m - 3) \geq 0$, which holds for all $m \neq 2$. Thus also $a_{m-1} \leq (m - 1)^2$ for all $m \neq 2$. If $m = 2$, then $a_2 \leq 2 \cdot 2 - 2 = 2$, but $a_2 \neq 2$ by our assumption, so $a_1 = a_2 \leq 1^2$.

Thus the three conditions are met for all $k \leq n$. Taking $k = 0$ implies $a_0 = 0$ which gives us an expression for x as a sum of distinct odd positive integers.

F24 (Grade 12.) Solve the equation

$$\sin 3x + \sin 7x = \sin 10x.$$

Answer: $x = \frac{1}{5}k\pi$ and $x = \frac{2}{3}k\pi$ and $x = \frac{2}{7}k\pi$, where k is an arbitrary integer.

Solution 1: Rewriting $10x = 3x + 7x$ and using the formula for the sine of the sum of two angles yields

$$\sin 3x + \sin 7x = \sin 3x \cos 7x + \cos 3x \sin 7x.$$

Factoring this yields

$$\sin 3x (1 - \cos 7x) + \sin 7x (1 - \cos 3x) = 0. \quad (45)$$

Writing $3x = 2 \cdot \frac{3}{2}x$ and $7x = 2 \cdot \frac{7}{2}x$, we can use the double angle sine and cosine formulas to get

$$\begin{aligned} \sin 3x (1 - \cos 7x) &= 2 \sin \frac{3}{2}x \cos \frac{3}{2}x (1 - \cos^2 \frac{7}{2}x + \sin^2 \frac{7}{2}x) \\ &= 4 \sin \frac{3}{2}x \cos \frac{3}{2}x \sin^2 \frac{7}{2}x. \end{aligned}$$

Similarly we get $\sin 7x (1 - \cos 3x) = 4 \sin \frac{7}{2}x \cos \frac{7}{2}x \sin^2 \frac{3}{2}x$. Thus Eq. (45) transforms to

$$4 \sin \frac{3}{2}x \cos \frac{3}{2}x \sin^2 \frac{7}{2}x + 4 \sin \frac{7}{2}x \cos \frac{7}{2}x \sin^2 \frac{3}{2}x = 0.$$

Dividing by 4 and factoring yields

$$\sin \frac{3}{2}x \sin \frac{7}{2}x \left(\sin \frac{7}{2}x \cos \frac{3}{2}x + \cos \frac{7}{2}x \sin \frac{3}{2}x \right) = 0,$$

where the formula for the sine of a sum of two angles gives

$$\sin \frac{3}{2}x \sin \frac{7}{2}x \sin 5x = 0.$$

This equation is equivalent to the original equation and holds if and only if any of the factors is equal to 0. Thus we get the solutions $x = \frac{2}{3}k\pi$, $x = \frac{2}{7}k\pi$ and $x = \frac{1}{5}k\pi$ where k is an arbitrary integer.

Solution 2: On the left hand side of the equation, we apply the formula $\sin \alpha + \sin \beta = 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}$ and on the right hand side, the double angle formula. This yields

$$2 \sin 5x \cos 2x = 2 \sin 5x \cos 5x.$$

Here $\sin 5x = 0$ holds for $x = \frac{1}{5}k\pi$, where k is an arbitrary integer. If $\sin 5x \neq 0$, then we can divide the sides by $2 \sin 5x$ to get $\cos 2x = \cos 5x$, which can be written as $\cos 2x - \cos 5x = 0$. Here, by applying the formula $\cos \alpha - \cos \beta = -2 \sin \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}$, we get

$$2 \sin \frac{7}{2}x \sin \frac{3}{2}x = 0.$$

Here $\sin \frac{7}{2}x = 0$ holds for $x = \frac{2}{7}k\pi$ and $\sin \frac{3}{2}x = 0$ holds for $x = \frac{2}{3}k\pi$.

Solution 3: We use complex numbers; let $w = \cos x + i \sin x$. Then

$$w^n = \cos nx + i \sin nx,$$

$$w^{-n} = \cos(-nx) + i \sin(-nx) = \cos nx - i \sin nx,$$

which enables us to express

$$\sin nx = \frac{w^n - w^{-n}}{2i}.$$

Now the equation given in the problem reduces to

$$\frac{w^3 - w^{-3}}{2i} + \frac{w^7 - w^{-7}}{2i} = \frac{w^{10} - w^{-10}}{2i}.$$

Multiplying both sides by $2i$ yields $w^3 - w^{-3} + w^7 - w^{-7} = w^{10} - w^{-10}$, which is equivalent to $w^{20} - w^{17} - w^{13} + w^7 + w^3 - 1 = 0$. Factoring yields

$$(w^{10} - 1)(w^7 - 1)(w^3 - 1) = 0.$$

Hence $w^3 = 1$ or $w^7 = 1$ or $w^{10} = 1$, which implies $x = \frac{2\pi n}{3}$ or $x = \frac{2\pi n}{7}$ or $x = \frac{2\pi n}{10}$ for any integer n .

Selected Problems from the IMO Team Selection Contests

S1 Let $x_1, x_2, \dots, x_n$ be positive real numbers, where $n \geq 2$. Prove that there exist indices i and j such that

$$x_i < \frac{(x_1 + x_2 + \dots + x_n)^2}{x_1 + 2x_2 + \dots + nx_n} < 2x_j.$$

Solution: Solving the problem is equivalent to showing that

$$x_i(x_1 + 2x_2 + \dots + nx_n) < (x_1 + x_2 + \dots + x_n)^2 < 2x_j(x_1 + 2x_2 + \dots + nx_n)$$

for some indices i and j .

Let $x_i = \min_{k=1, \dots, n} x_k$. Note that, for every $k = 1, 2, \dots, n$,

$$kx_i = \underbrace{x_i + x_i + \dots + x_i}_{k \text{ addends}} \leq x_1 + x_2 + \dots + x_k.$$

Thus

$$\begin{aligned} x_i(x_1 + 2x_2 + \dots + nx_n) &= x_i x_1 + 2x_i x_2 + \dots + nx_i x_n \\ &\leq x_1 x_1 + (x_1 + x_2)x_2 + \dots + (x_1 + x_2 + \dots + x_n)x_n \\ &= (x_1^2 + x_2^2 + \dots + x_n^2) + (x_1 x_2 + x_1 x_3 + \dots + x_{n-1} x_n) \\ &< (x_1^2 + x_2^2 + \dots + x_n^2) + 2(x_1 x_2 + x_1 x_3 + \dots + x_{n-1} x_n) \\ &= (x_1 + x_2 + \dots + x_n)^2. \end{aligned}$$

Let $x_j = \max_{k=1, \dots, n} x_k$. Like before, note that, for every $k = 1, 2, \dots, n$,

$$kx_j = \underbrace{x_j + x_j + \dots + x_j}_{k \text{ addends}} \geq x_1 + x_2 + \dots + x_k.$$

Thus

$$\begin{aligned}
2x_j(x_1 + 2x_2 + \dots + nx_n) &= 2(x_jx_1 + 2x_jx_2 + \dots + nx_jx_n) \\
&\geq 2(x_1x_1 + (x_1 + x_2)x_2 + \dots + (x_1 + x_2 + \dots + x_n)x_n) \\
&= 2(x_1^2 + x_2^2 + \dots + x_n^2) + 2(x_1x_2 + x_1x_3 + \dots + x_{n-1}x_n) \\
&> (x_1^2 + x_2^2 + \dots + x_n^2) + 2(x_1x_2 + x_1x_3 + \dots + x_{n-1}x_n) \\
&= (x_1 + x_2 + \dots + x_n)^2.
\end{aligned}$$

S2 Let $\mathbb{R}$ be the set of real numbers. For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and for all real numbers x, y and z the following inequality holds:

$$f(x) + f(y) + f(z) \geq f(xy) + f(yz) + f(zx).$$

Prove that f is an even function.

Solution 1: Denote the given inequality by $P(x, y, z)$. Then $P(-1, 1, 1)$ gives $f(-1) + f(1) + f(1) \geq f(-1) + f(1) + f(-1)$, implying $f(1) \geq f(-1)$. On the other hand, $P(-1, -1, -1)$ gives $f(-1) + f(-1) + f(-1) \geq f(1) + f(1) + f(1)$, reducing to $f(-1) \geq f(1)$. Altogether, we have proven that

$$f(1) = f(-1). \quad (46)$$

Let now x be an arbitrary non-zero real number. Then $P\left(x, \frac{1}{x}, -1\right)$ implies $f(x) + f\left(\frac{1}{x}\right) + f(-1) \geq f(1) + f\left(-\frac{1}{x}\right) + f(-x)$. By applying Eq. (46) and rearranging terms, we obtain

$$f(x) - f\left(-\frac{1}{x}\right) \geq f(-x) - f\left(\frac{1}{x}\right). \quad (47)$$

On the other hand, $P\left(x, -\frac{1}{x}, -1\right)$ along with cancelling of like terms yields

$$f(x) + f\left(-\frac{1}{x}\right) \geq f\left(\frac{1}{x}\right) + f(-x). \quad (48)$$

Adding the inequalities (47) and (48) along with division by 2 leads to $f(x) \geq f(-x)$. As this inequality holds for every non-zero real number x , also $f(-x) \geq f(-(-x)) = f(x)$ holds for every non-zero real number x . Altogether, we have proven the equality $f(x) = f(-x)$ for every non-zero real number x . This means that f is an even function.

Solution 2: Like in Solution 1, denote the given inequality by $P(x, y, z)$. Denote $f(0) = a$ and $f(1) = b$. For any real number x , $P(x, 0, 0)$ gives $f(x) + a + a \geq a + a + a$, implying

$$f(x) \geq a \quad (49)$$

for all x . On the other hand, $P(x, 1, 1)$ gives $f(x) + b + b \geq f(x) + b + f(x)$, implying

$$b \geq f(x) \quad (50)$$

for all x . Hence all values of f are bounded between a and b .

Next, let x be an arbitrary non-zero real number. Then $P\left(x, x, \frac{1}{x}\right)$ along with (50) gives $f(x) + f(x) + f\left(\frac{1}{x}\right) \geq f(x^2) + b + b \geq f(x^2) + f(x) + b$, which simplifies to

$$f(x) + f\left(\frac{1}{x}\right) \geq f(x^2) + b. \quad (51)$$

Denote (51) by $Q(x)$. As b is an upper bound of all values of f , $Q(x)$ implies $f(x) + f\left(\frac{1}{x}\right) \geq f(x^2) + f\left(\frac{1}{x}\right)$ and, in a similar fashion, $Q\left(\frac{1}{x}\right)$ leads to $f\left(\frac{1}{x}\right) + f(x) \geq f\left(\frac{1}{x^2}\right) + f\left(\frac{1}{x}\right)$. These inequalities simplify to

$$\begin{aligned} f(x) &\geq f(x^2), \\ f(x) &\geq f\left(\frac{1}{x^2}\right), \end{aligned} \quad (52)$$

respectively.

We now prove that $f(x) = b$ for every non-zero x . Suppose the contrary, i.e., $f(x) < b$ for some non-zero x . As there is a lower bound a of all values of f , there also must exist a constant

$$c = \inf_{x \neq 0} f(x). \quad (53)$$

By our assumption, $c < b$, implying that we can find a non-zero x such that $f(x) < \frac{c+b}{2}$. Using (52) for that x , we obtain

$$f(x^2) + f\left(\frac{1}{x^2}\right) \leq f(x) + f(x) = 2f(x) < 2 \cdot \frac{c+b}{2} = c+b.$$

On the other hand, $Q(x^2)$ along with (53) gives

$$f(x^2) + f\left(\frac{1}{x^2}\right) \geq f(x^4) + b \geq c+b.$$

The contradiction establishes that indeed $f(x) = b$ for all non-zero x .

Hence $f(x) = b = f(-x)$ for all non-zero x , so f is even as desired.

Remark: Solution 2 shows that all functions that satisfy the inequality of the problem are of the form

$$f(x) = \begin{cases} b & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}$$

where a and b are arbitrary real numbers such that $a \leq b$.

S3 Find all integers $n > 2$ for which there exists a set $\mathcal{P}$ of n points in the plane, such that whenever A and B are distinct points in $\mathcal{P}$, then there exists a point C in $\mathcal{P}$, such that ABC is equilateral.

Answer: 3.

Solution 1: For $n = 3$, three points forming an equilateral triangle satisfy the conditions. In the following we assume that $n > 3$.

Let A and B be two points in $\mathcal{P}$ with the maximal distance d between them. Then there exists $C \in \mathcal{P}$ such that $AC = BC = AB = d$.

We will show that in the set $\mathcal{P} \setminus \{A, B, C\}$ there exists a point which is not in the interior of ABC (being on its sides is allowed). Assume for contradiction that all points from the set $\mathcal{P} \setminus \{A, B, C\}$ lie in the interior of ABC . Let D be any of them. Then there exists a point E such that ADE is equilateral. By our assumption, E lies in the interior of ABC as well (Fig. 51). But then we have $\angle DAE < 60^\circ$, contradiction. So indeed there must exist a point in $\mathcal{P} \setminus \{A, B, C\}$ which is not in the interior of ABC .

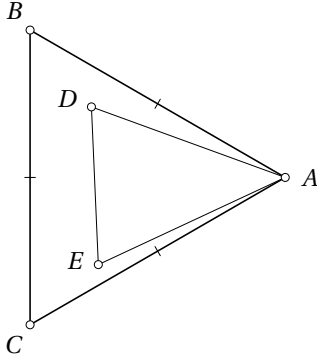


Fig. 51

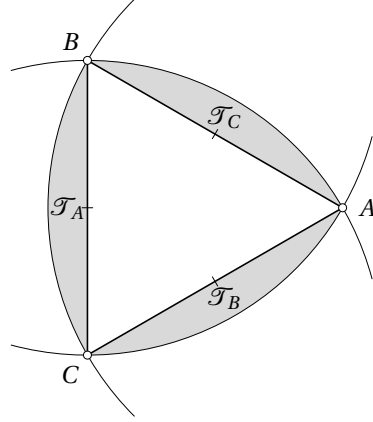


Fig. 52

Let $\mathcal{T}_A$ be the set of points P for which $AP \leq d$ and the segments AP and BC intersect. Define $\mathcal{T}_B$ and $\mathcal{T}_C$ analogously (Fig. 52). Each point $X \in \mathcal{P}$ not located in the interior of ABC must therefore be in one of these sets, otherwise one of the segments AX , BX and CX would be longer than d .

Assume without loss of generality that $D \in (\mathcal{P} \setminus \{A, B, C\}) \cap \mathcal{T}_A$ for some point D . Then there exists a point $E \in \mathcal{P}$ for which ADE is equilateral. If $E \in \mathcal{T}_A$ or E lies in the interior of ABC , then both D and E lie within the angle BAC (Fig. 53). But then $\angle DAE < 60^\circ$, contradiction. So E must lie in $\mathcal{T}_B$ or $\mathcal{T}_C$. Assume without loss of generality that $E \in \mathcal{T}_B$ (Fig. 54).

Now we notice that

$$\angle BAE = \angle BAD + \angle DAE = \angle BAD + 60^\circ > 60^\circ$$

and

$$\angle AEB = \angle AED - \angle BED = 60^\circ - \angle BED < 60^\circ.$$

Thus $\angle BAE > \angle AEB$, which means $BE > AB = d$, contradicting the definition of d . Thus $n > 3$ is not possible.

Solution 2: For $n = 3$, three points forming an equilateral triangle satisfy the conditions. In the following we assume that $n > 3$.

Let A and B be two points in $\mathcal{P}$ with the maximal distance d between them. Then there exists $C \in \mathcal{P}$ such that $AC = BC = AB = d$. Consider the three

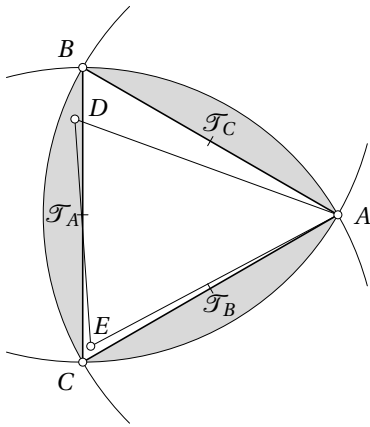


Fig. 53

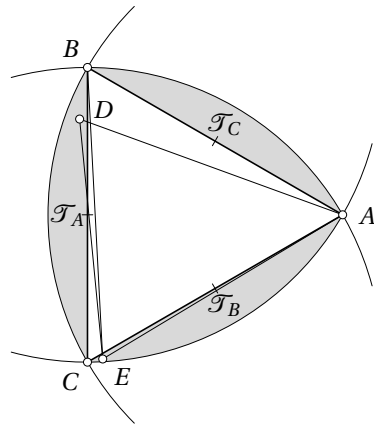


Fig. 54

circles ω_A , ω_B and ω_C , each with radius d , centered at A , B and C respectively. By the definition of d , all points in $\mathcal{P}$ must lie in the intersection of these circles (Fig. 55), denoted by $\mathcal{U}$. Define $\mathcal{V}_B$ as the shape obtained by rotating $\mathcal{U}$ by 60° around A such that B goes to C , and taking the intersection of the original and new $\mathcal{U}$. Define $\mathcal{V}_C$ and $\mathcal{V}_A$ analogously (Fig. 56).

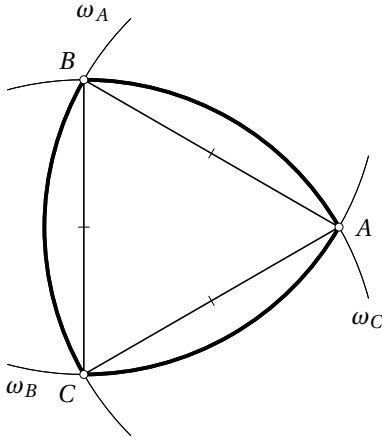


Fig. 55

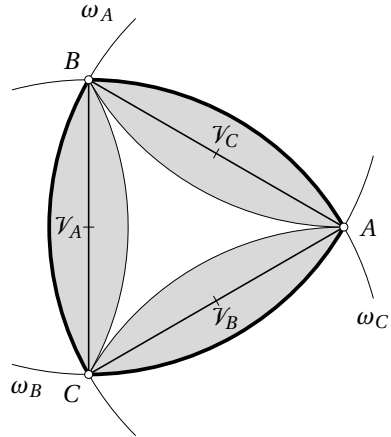


Fig. 56

Consider any $D \in \mathcal{P} \setminus \{A, B, C\}$. Assume without loss of generality that D lies on the same side of the perpendicular bisector of BC as B (or on the bisector itself). Then there exists $E \in \mathcal{P}$, such that ADE is equilateral. Consider first the case where E is obtained from D by the rotation centered at A which also takes C to B (Fig. 57). Then

$$\angle EAC = \angle EAD + \angle DAC = 60^\circ + \angle DAC \geq 60^\circ + 30^\circ = 90^\circ.$$

